HL Paper 2

Boxes of mixed fruit are on sale at a local supermarket.

Box A contains 2 bananas, 3 kiwifruit and 4 melons, and costs $6.58.

Box B contains 5 bananas, 2 kiwifruit and 8 melons and costs $12.32.

Box C contains 5 bananas and 4 kiwifruit and costs $3.00.

Find the cost of each type of fruit.

Markscheme

let b be the cost of one banana, k the cost of one kiwifruit, and m the cost of one melon

attempt to set up three linear equations ***(M1)***

$2b + 3k + 4m = 658$

$5b + 2k + 8m = 1232$

$5b + 4k = 300$ ***(A1)***

attempt to solve three simultaneous equations ***(M1)***

$b = 36,\ k = 30,\ m = 124$

banana costs ($)0.36, kiwifruit costs ($)0.30, melon costs ($)1.24 ***A1***

[4 marks]

Examiners report

[N/A]

Given that $\log_{10}\left(\frac{1}{2\sqrt{2}}(p+2q)\right) = \frac{1}{2}(\log_{10}p + \log_{10}q)$, $p > 0,\ q > 0$, find p in terms of q.

Markscheme

$\log_{10}\frac{1}{2\sqrt{2}}(p+2q) = \frac{1}{2}(\log_{10}p + \log_{10}q)$

$\log_{10}\frac{1}{2\sqrt{2}}(p+2q) = \frac{1}{2}\log_{10}pq$ ***(M1)***

$\log_{10}\frac{1}{2\sqrt{2}}(p+2q) = \log_{10}(pq)^{\frac{1}{2}}$ ***(M1)***

$\frac{1}{2\sqrt{2}}(p+2q) = (pq)^{\frac{1}{2}}$ ***(A1)***

$(p+2q)^2 = 8pq$

$p^2 + 4pq + 4q^2 = 8pq$

$p^2 - 4pq + 4q^2 = 0$

$(p-2q)^2=0$ ***M1***

hence $p=2q$ ***A1***

[5 marks]

Examiners report

[N/A]

Use the method of mathematical induction to prove that $5^{2n}-24n-1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

Markscheme

$\mathrm{P}(n): f(n)=5^{2n}-24n-1$ is divisible by 576 for $n \in \mathbb{Z}^+$

for $n=1,\ f(1)=5^2-24-1=0$

Zero is divisible by 576, (as every non-zero number divides zero), and so P(1) is true. ***R1***

Note: Award ***R0*** for P(1) = 0 shown and zero is divisible by 576 not specified.

Note: Ignore P(2) = 576 if P(1) = 0 is shown and zero is divisible by 576 is specified.

Assume $\mathrm{P}(k)$ is true for some k $(\Rightarrow f(k)=N\times 576)$. ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1): f(k+1)=5^{2(k+1)}-24(k+1)-1$ ***M1***

$=25\times 5^{2k}-24k-25$ ***A1***

EITHER

$=25\times(24k+1+N\times 576)-24k-25$ ***A1***

$=576k+25\times 576N$ which is a multiple of 576 ***A1***

OR

$=25\times 5^{2k}-600k-25+600k-24k$ ***A1***

$=25(5^{2k}-24k-1)+576k$ (or equivalent) which is a multiple of 576 ***A1***

THEN

P(1) is true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, so $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award ***R1*** only if at least four prior marks have been awarded.

[7 marks]

Examiners report

This proof by mathematical induction challenged most candidates. While most candidates were able to show that P(1) = 0, a significant number did not state that zero is divisible by 576. A few candidates started their proof by looking at P(2). It was pleasing to see that the inductive step was reasonably well done by most candidates. However many candidates committed simple algebraic errors. The most common error was to state that $5^{2(k+1)} = 5(5)^{2k}$. The concluding statement often omitted the required implication statement and also often omitted that P(1) was found to be true.

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a. Show that $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = 1$. [1]

b. Consider the geometric series $1 + \frac{1}{3}\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{9}\mathrm{e}^{2\mathrm{i}\theta} + \ldots$. [2]

Write down the common ratio, z, of the series, and show that $|z| = \frac{1}{3}$.

c. Find an expression for the sum to infinity of this series. [2]

d. Hence, show that $\sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \ldots = \frac{9\sin\theta}{10 - 6\cos\theta}$. [8]

Markscheme

a. $\left|\mathrm{e}^{\mathrm{i}\theta}\right| \;\left(= |\cos\theta + \mathrm{i}\sin\theta|\right) = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ ***M1AG***

[1 mark]

b. $z = \frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}$ ***A1***

$|z| = \left|\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}\right| = \frac{1}{3}$ ***A1AG***

[2 marks]

c. $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}}$ ***(M1)A1***

[2 marks]

d. **EITHER**

$S_\infty = \frac{1}{1-\frac{1}{3}\cos\theta-\frac{1}{3}\mathrm{i}\sin\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta+\frac{1}{3}\mathrm{i}\sin\theta}{\left(1-\frac{1}{3}\cos\theta-\frac{1}{3}\mathrm{i}\sin\theta\right)\left(1-\frac{1}{3}\cos\theta+\frac{1}{3}\mathrm{i}\sin\theta\right)}$ ***M1A1***

$= \frac{1-\frac{1}{3}\cos\theta+\frac{1}{3}\mathrm{i}\sin\theta}{\left(1-\frac{1}{3}\cos\theta\right)^2+\frac{1}{9}\sin^2\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta+\frac{1}{3}\mathrm{i}\sin\theta}{1-\frac{2}{3}\cos\theta+\frac{1}{9}}$ ***A1***

OR

$S_\infty = \frac{1}{1-\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}}$

$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\left(1-\frac{1}{3}\mathrm{e}^{\mathrm{i}\theta}\right)\left(1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}\right)}$ ***M1A1***

$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{1-\frac{1}{3}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+\frac{1}{9}}$ ***A1***

$= \frac{1-\frac{1}{3}\mathrm{e}^{-\mathrm{i}\theta}}{\frac{10}{9}-\frac{2}{3}\cos\theta}$ ***A1***

$= \frac{1 - \frac{1}{3}(\cos\theta - \mathrm{i}\sin\theta)}{\frac{10}{9} - \frac{2}{3}\cos\theta}$ ***A1***

THEN

taking imaginary parts on both sides

$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \ldots = \frac{\frac{1}{3}\sin\theta}{\frac{10}{9} - \frac{2}{3}\cos\theta}$ ***M1A1A1***

$= \frac{\sin\theta}{\frac{10}{9} - \frac{2}{3}\cos\theta}$

$\Rightarrow \sin\theta + \frac{1}{3}\sin 2\theta + \ldots = \frac{9\sin\theta}{10 - 6\cos\theta}$ ***AG***

[8 marks]

Examiners report

a. Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

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Consider the system of equations

$$0.1x - 1.7y + 0.9z = -4.4$$

$$-2.4x + 0.3y + 3.2z = 1.2$$

$$2.5x + 0.6y - 3.7z = 0.8.$$

a. Express the system of equations in matrix form. [2]

b. Find the solution to the system of equations. [3]

Markscheme

a. attempting to express the system in matrix form ***M1***

$$\begin{pmatrix} 0.1 & -1.7 & 0.9 \\ -2.4 & 0.3 & 3.2 \\ 2.5 & 0.6 & -3.7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4.4 \\ 1.2 \\ 0.8 \end{pmatrix} \quad \textbf{\textit{A1}}$$

Note: Award ***M1A1*** for a correct augmented matrix.

[2 marks]

b. either direct GDC use, attempting elimination or using an inverse matrix. ***(M1)***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2.4 \\ 1.6 \\ -1.6 \end{pmatrix} \text{ (correct to 2sf) or } \begin{pmatrix} -2.40 \\ 1.61 \\ -1.57 \end{pmatrix} \text{ (correct to 3sf) or } \begin{pmatrix} -\frac{932}{389} \\ \frac{628}{389} \\ -\frac{612}{389} \end{pmatrix} \text{ (exact)} \quad \textbf{\textit{A2}}$$

[3 marks]

Examiners report

a. This was generally well done. In part (a), some candidates expressed the system of equations in the form $XA = B$. In part (b), the overwhelming majority of candidates who used a direct GDC approach obtained the correct solution. Candidates who attempted matrix methods such as row reduction without a GDC were generally unsuccessful.

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(a) Solve the equation $z^3 = -2 + 2\mathrm{i}$, giving your answers in modulus-argument form.

(b) **Hence** show that one of the solutions is $1 + \mathrm{i}$ when written in Cartesian form.

Markscheme

(a) $z^3 = 2\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}$ ***(M1)(A1)***

$z_1 = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$ ***A1***

adding or subtracting $\frac{2\pi\mathrm{i}}{3}$ ***M1***

$z_2 = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}+\frac{2\pi\mathrm{i}}{3}} = \sqrt{2}\mathrm{e}^{\frac{11\pi\mathrm{i}}{12}}$ ***A1***

$z_3 = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}-\frac{2\pi\mathrm{i}}{3}} = \sqrt{2}\mathrm{e}^{-\frac{5\pi\mathrm{i}}{12}}$ ***A1***

Notes: Accept equivalent solutions *e.g.* $z_3 = \sqrt{2}\mathrm{e}^{\frac{19\pi\mathrm{i}}{12}}$

Award marks as appropriate for solving $(a + b\mathrm{i})^3 = -2 + 2\mathrm{i}$.

Accept answers in degrees.

(b) $\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}} \left(= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}\right)\right)$ ***A1***

$= 1 + \mathrm{i}$ ***AG***

Note: Accept geometrical reasoning.

[7 marks]

Examiners report

Many students incorrectly found the argument of z^3 to be $\arctan\left(\frac{2}{-2}\right) = -\frac{\pi}{4}$. Of those students correctly finding one solution, many were unable to use symmetry around the origin, to find the other two. In part (b) many students found the cube of 1 + *i* which could not be awarded marks as it was not "hence".

Six people are to sit at a circular table. Two of the people are not to sit immediately beside each other. Find the number of ways that the six people can be seated.

Markscheme

EITHER

with no restrictions six people can be seated in $5! = 120$ ways ***A1***

we now count the number of ways in which the two restricted people will be sitting next to each other

call the two restricted people p_1 and p_2

they sit next to each other in two ways ***A1***

the remaining people can then be seated in 4! ways ***A1***

the six may be seated p_1 and p_2 next to each other) in $2 \times 4! = 48$ ways ***M1***

$\therefore$ with p_1 and (p_2 not next to each other the number of ways $= 120 - 48 = 72$ ***A1*** ***N3***

[5 marks]

OR

person p_1 seated at table in 1 way ***A1***

p_2 then sits in any of 3 seats (not next to p_1) ***M1A1***

the remaining 4 people can then be seated in 4! ways ***A1***

$\therefore$ number ways with p_1 not next to $p_2 = 3 \times 4! = 72$ ways ***A1*** ***N3***

Note: If candidate starts with 6! instead of 5!, potentially leading to an answer of 432, do not penalise.

[5 marks]

Examiners report

Very few candidates provided evidence of a clear strategy for solving such a question. The problem which was set in a circular scenario was no more difficult than an analogous linear one.

A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece.

Markscheme

the pieces have lengths $a,\ ar,\ ...,\ ar^9$ ***(M1)***

$8a = ar^9$ (or $8 = r^9$) ***A1***

$r = \sqrt[9]{8} = 1.259922...$ ***A1***

$a\frac{r^{10}-1}{r-1} = 1 \quad \left(\text{or } a\frac{r^{10}-1}{r-1} = 1000\right)$ ***M1***

$a = \frac{r-1}{r^{10}-1} = 0.0286... \quad \left(\text{or } a = \frac{r-1}{r^{10}-1} = 28.6...\right)$ ***(A1)***

a = 29 mm (accept 0.029 m or any correct answer regardless the units) ***A1***

[6 marks]

Examiners report

This question was generally well done by most candidates. Some candidates recurred to a diagram to comprehend the nature of the problem but a few thought it was an arithmetic sequence.

A surprising number of candidates missed earning the final A1 mark because they did not read the question instructions fully and missed the accuracy instruction to give the answer correct to the nearest mm.

Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}(1-\cos\theta))^2$ where θ is an acute angle.

Markscheme

$(\sin\theta + \mathrm{i}(1-\cos\theta))^2 = \sin^2\theta - (1-\cos\theta)^2 + \mathrm{i}2\sin\theta(1-\cos\theta)$ ***M1A1***

Let α be the required argument.

$\tan\alpha = \frac{2\sin\theta(1-\cos\theta)}{\sin^2\theta-(1-\cos\theta)^2}$ ***M1***

$= \frac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta)-(1-2\cos\theta+\cos^2\theta)}$ ***(M1)***

$= \frac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ ***A1***

$= \tan\theta$ ***A1***

$\alpha = \theta$ ***A1***

[7 marks]

Examiners report

Very few candidates scored more than the first two marks in this question. Some candidates had difficulty manipulating trigonometric identities.

Most candidates did not get as far as defining the argument of the complex expression.

The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1 z_2 = -\sqrt{3} + \mathrm{i}$ and $\frac{z_1}{z_2} = 2\mathrm{i}$, find the modulus and argument of z_1 and of z_2.

Markscheme

METHOD 1

$\arg(z_1 z_2) = \frac{5\pi}{6} \quad (150^\circ)$ ***(A1)***

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \quad (90^\circ)$ ***(A1)***

$\Rightarrow \arg(z_1) + \arg(z_2) = \frac{5\pi}{6};\ \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$ ***M1***

solving simultaneously

$\arg(z_1) = \frac{2\pi}{3}\ (120^\circ)$ and $\arg(z_2) = \frac{\pi}{6}\ (30^\circ)$ ***A1A1***

Note: Accept decimal approximations of the radian measures.

$|z_1 z_2| = 2 \Rightarrow |z_1|\,|z_2| = 2;\ \left|\frac{z_1}{z_2}\right| = 2 \Rightarrow \frac{|z_1|}{|z_2|} = 2$ ***M1***

solving simultaneously

$|z_1| = 2;\ |z_2| = 1$ ***A1***

[7 marks]

METHOD 2

$z_1 = 2iz_2 \quad 2iz_2^2 = -\sqrt{3} + i$ ***(M1)***

$z_2^2 = \frac{-\sqrt{3}+i}{2i}$ ***A1***

$z_2 = \sqrt{\frac{-\sqrt{3}+i}{2i}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $e^{\frac{\pi}{6}i}$ ***(M1)(A1)***

(allow $0.866 + 0.5i$ or $\mathrm{e}^{0.524\mathrm{i}}$)

$z_1 = -1 + \sqrt{3}i$ or $2\mathrm{e}^{\frac{2\pi}{3}i}$– (allow $-1 + 1.73\mathrm{i}$ or $2\mathrm{e}^{2.09\mathrm{i}}$) ***(A1)***

z_1 modulus = 2, argument $= \frac{2\pi}{3}$ ***A1***

z_2 modulus = 1, argument $= \frac{\pi}{6}$ ***A1***

Note: Accept degrees and decimal approximations to radian measure.

[7 marks]

Examiners report

Candidates generally found this question challenging. Many candidates had difficulty finding the arguments of $z_1 z_2$ and z_1/z_2. Among candidates who attempted to solve for z_1 and z_2 in Cartesian form, many had difficulty with the algebraic manipulation involved.

Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

a. Find the probability that Ava wins on her first turn. [1]

b. Find the probability that Barry wins on his first turn. [2]

c. Find the probability that Ava wins in one of her first three turns. [4]

d. Find the probability that Ava eventually wins. [4]

Markscheme

a. $\mathrm{P(Ava\ wins\ on\ her\ first\ turn)} = \frac{1}{3}$ ***A1***

[1 mark]

b. $\mathrm{P(Barry\ wins\ on\ his\ first\ turn)} = \left(\frac{2}{3}\right)^2$ ***(M1)***

$= \frac{4}{9} \quad (= 0.444)$ ***A1***

[2 marks]

c. P(Ava wins in one of her first three turns)

$= \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3}$ ***M1A1A1***

Note: Award ***M1*** for adding probabilities, award ***A1*** for a correct second term and award ***A1*** for a correct third term.

Accept a correctly labelled tree diagram, awarding marks as above.

$= \frac{103}{243} \quad (= 0.424)$ ***A1***

[4 marks]

d. $\mathrm{P(Ava\ eventually\ wins)} = \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{1}{3} + \dots$ ***(A1)***

using $S_\infty = \frac{a}{1-r}$ with $a = \frac{1}{3}$ and $r = \frac{2}{9}$ ***(M1)(A1)***

Note: Award ***(M1)*** for using $S_\infty = \frac{a}{1-r}$ and award ***(A1)*** for $a = \frac{1}{3}$ and $r = \frac{2}{9}$.

$= \frac{3}{7} \quad (= 0.429)$ ***A1***

[4 marks]

Total [11 marks]

Examiners report

a. Parts (a) and (b) were straightforward and were well done.

b. Parts (a) and (b) were straightforward and were well done.

c. Parts (c) and (d) were also reasonably well done.

d. Parts (c) and (d) were also reasonably well done. A pleasingly large number of candidates recognized that an infinite geometric series was required in part (d).

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

a. (i) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$. [6]

(ii) Hence use De Moivre's theorem to prove

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

(iii) State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

b. Find the value of r and the value of α. [4]

c. Using (a) (ii) and your answer from (b) show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$. [4]

d. Hence express $\sin 72^\circ$ in the form $\frac{\sqrt{a+b\sqrt{c}}}{d}$ where $a,\ b,\ c,\ d \in \mathbb{Z}$. [5]

Markscheme

a. (i) $(\cos\theta + \mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta +$

$10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$ ***A1A1***

$(= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta -$

$10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta)$

Note: Award first ***A1*** for correct binomial coefficients.

(ii) $(\mathrm{cis}\theta)^5 = \mathrm{cis}5\theta = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***M1***

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta +$

$5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$ ***A1***

Note: Previous line may be seen in (i)

equating imaginary terms ***M1***

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ***AG***

(iii) equating real terms

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***A1***

[6 marks]

b. $(r\text{cis}\alpha)^5 = 1 \Rightarrow r^5\text{cis}5\alpha = 1\text{cis}0$ ***M1***

$r^5 = 1 \Rightarrow r = 1$ ***A1***

$5\alpha = 0 \pm 360k,\ k \in \mathbb{Z} \Rightarrow a = 72k$ ***(M1)***

$\alpha = 72^\circ$ ***A1***

Note: Award ***M1A0*** if final answer is given in radians.

[4 marks]

c. use of $\sin(5 \times 72) = 0$ **OR** the imaginary part of 1 is 0 ***(M1)***

$0 = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$ ***A1***

$\sin\alpha \neq 0 \Rightarrow 0 = 5(1 - \sin^2\alpha)^2 - 10(1 - \sin^2\alpha)\sin^2\alpha + \sin^4\alpha$ ***M1***

Note: Award ***M1*** for replacing $\cos^2\alpha$.

$0 = 5(1 - 2\sin^2\alpha + \sin^4\alpha) - 10\sin^2\alpha + 10\sin^4\alpha + \sin^4\alpha$ ***A1***

Note: Award ***A1*** for any correct simplification.

so $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$ ***AG***

[4 marks]

d. $\sin^2\alpha = \frac{20 \pm \sqrt{400 - 320}}{32}$ ***M1A1***

$\sin\alpha = \pm\sqrt{\frac{20 \pm \sqrt{80}}{32}}$

$\sin\alpha = \frac{\pm\sqrt{10 \pm 2\sqrt{5}}}{4}$ ***A1***

Note: Award ***A1*** regardless of signs. Accept equivalent forms with integral denominator, simplification may be seen later.

as $72 > 60$, $\sin 72 > \frac{\sqrt{3}}{2} = 0.866\ldots$ we have to take both positive signs (or equivalent argument) ***R1***

Note: Allow verification of correct signs with calculator if clearly stated

$\sin 72 = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$ ***A1***

[5 marks]

Total [19 marks]

Examiners report

a. In part (i) many candidates tried to multiply it out the binomials rather than using the binomial theorem. In parts (ii) and (iii) many candidates showed poor understanding of complex numbers and made no attempt to equate real and imaginary parts. In a some cases the correct answer to part (iii) was seen although it was unclear how it was obtained.

b. This question was poorly done. Very few candidates made a good attempt to apply De Moivre's theorem and most of them could not even equate the moduli to obtain r.

c. This question was poorly done. From the few candidates that attempted it, many candidates started by writing down what they were trying to prove and made no progress.

d. Very few made a serious attempt to answer this question. Also very few realised that they could use the answers given in part (c) to attempt this part.

Determine the first three terms in the expansion of $(1-2x)^5(1+x)^7$ in ascending powers of x.

Markscheme

METHOD 1

constant term: $\binom{5}{0}(-2x)^0\binom{7}{0}x^0 = 1$ ***A1***

term in x: $\binom{7}{1}x + \binom{5}{1}(-2x) = -3x$ ***(M1)A1***

term in x^2 : $\binom{7}{2}x^2 + \binom{5}{2}(-2x)^2 + \binom{7}{1}x\binom{5}{1}(-2x) = -9x^2$ ***M1A1*** ***N3***

[5 marks]

METHOD 2

$(1-2x)^5(1+x)^7 = \left(1+5(-2x)+\frac{5\times4(-2x)^2}{2!}+\ldots\right)\left(1+7x+\frac{7\times6}{2}x^2+\ldots\right)$ ***M1M1***

$= (1-10x+40x^2+\ldots)(1+7x+21x^2+\ldots)$

$= 1+7x+21x^2-10x-70x^2+40x^2+\ldots$

$= 1-3x-9x^2+\ldots$ ***A1A1A1*** ***N3***

[5 marks]

Examiners report

Although the majority of the candidates understood the question and attempted it, excessive time was spent on actually expanding the expression without consideration of the binomial theorem. A fair amount of students confused "ascending order", giving the last three instead of the first three terms.

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

a. Show that $d = \frac{a}{2}$. [3]

b. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. [6]

Find the least value of n for which this occurs.

Markscheme

a. using $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$ to form $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$ ***(M1)***

$a(a+6d) = (a+2d)^2$ ***A1***

$2d(2d-a) = 0$ (or equivalent) ***A1***

since $d \neq 0 \Rightarrow d = \frac{a}{2}$ ***AG***

[3 marks]

b. substituting $d = \frac{a}{2}$ into $a + 6d = 3$ and solving for a and d ***(M1)***

$a = \frac{3}{4}$ and $d = \frac{3}{8}$ ***(A1)***

$r = \frac{1}{2}$ ***A1***

$\frac{n}{2}\left(2 \times \frac{3}{4} + (n-1)\frac{3}{8}\right) - \frac{3\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} \geq 200$ ***(A1)***

attempting to solve for n ***(M1)***

$n \geq 31.68\ldots$

so the least value of n is 32 ***A1***

[6 marks]

Total [9 marks]

Examiners report

a. Part (a) was reasonably well done. A number of candidates used $r = \frac{u_1}{u_2} = \frac{u_2}{u_3}$ rather than $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$. This invariably led to candidates obtaining $r = 2$ in part (b).

b. In part (b), most candidates were able to correctly find the first term and the common difference for the arithmetic sequence. However a number of candidates either obtained $r = 2$ via means described in part (a) or confused the two sequences and used $u_1 = \frac{3}{4}$ for the geometric sequence.

The arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$.

a. Find an expression for $u_n - v_n$ in terms of n. [2]

b. Determine the set of values of n for which $u_n > v_n$. [3]

c. Determine the greatest value of $u_n - v_n$. Give your answer correct to four significant figures. [1]

Markscheme

a. $u_n - v_n = 1.6 + (n-1) \times 1.5 - 3 \times 1.2^{n-1}$ $(= 1.5n + 0.1 - 3 \times 1.2^{n-1})$ ***A1A1***

[2 marks]

b. attempting to solve $u_n > v_n$ numerically or graphically. ***(M1)***

$n = 2.621\ldots, 9.695\ldots$ ***(A1)***

So $3 \leqslant n \leqslant 9$ ***A1***

[3 marks]

c. The greatest value of $u_n - v_n$ is 1.642. ***A1***

Note: Do not accept 1.64.

[1 mark]

Examiners report

a. In part (a), most candidates were able to express u_n and v_n correctly and hence obtain a correct expression for $u_n - v_n$. Some candidates made careless algebraic errors when unnecessarily simplifying u_n while other candidates incorrectly stated v_n as $3(1.2)^n$.

b. In parts (b) and (c), most candidates treated n as a continuous variable rather than as a discrete variable. Candidates should be aware that a GDC's table feature can be extremely useful when attempting such question types.

c. In parts (b) and (c), most candidates treated n as a continuous variable rather than as a discrete variable. Candidates should be aware that a GDC's table feature can be extremely useful when attempting such question types. In part (c), a number of candidates attempted to find the maximum value of n rather than attempting to find the maximum value of $u_n - v_n$.

Each time a ball bounces, it reaches 95 % of the height reached on the previous bounce.
Initially, it is dropped from a height of 4 metres.

a. What height does the ball reach after its fourth bounce? [2]

b. How many times does the ball bounce before it no longer reaches a height of 1 metre? [3]

c. What is the total distance travelled by the ball? [3]

Markscheme

a. height $= 4 \times 0.95^4$ ***(A1)***

$= 3.26$ (metres) ***A1***

[2 marks]

b. $4 \times 0.95^n < 1$ ***(M1)***

$0.95^n < 0.25$

$\Rightarrow n > \frac{\ln 0.25}{\ln 0.95}$ ***(A1)***

$\Rightarrow n > 27.0$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ ***A1***

Note: If candidates have used $n - 1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

[3 marks]

c. **METHOD 1**

recognition of geometric series with sum to infinity, first term of 4×0.95 and common ratio 0.95 ***M1***

recognition of the need to double this series and to add 4 ***M1***

total distance travelled is $2\left(\frac{4 \times 0.95}{1 - 0.95}\right) + 4 = 156$ (metres) ***A1***

[3 marks]

Note: If candidates have used $n - 1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

METHOD 2

recognition of a geometric series with sum to infinity, first term of 4 and common ratio 0.95 **M1**

recognition of the need to double this series and to subtract 4 **M1**

total distance travelled is $2\left(\frac{4}{1 - 0.95}\right) - 4 = 156$ (metres) **A1**

[3 marks]

Examiners report

a. The majority of candidates were able to start this question and gain some marks, but only better candidates gained full marks. In part (a) the common error was to assume the wrong number of bounces and in part (b) many candidates lost marks due to rounding the inequality in the wrong direction. Part (c) was found difficult with only a limited number recognising the need for the sum to infinity of a geometric sequence and many of those did not recognise how to link the sum to infinity to the total distance travelled.

b. The majority of candidates were able to start this question and gain some marks, but only better candidates gained full marks. In part (a) the common error was to assume the wrong number of bounces and in part (b) many candidates lost marks due to rounding the inequality in the wrong direction. Part (c) was found difficult with only a limited number recognising the need for the sum to infinity of a geometric sequence and many of those did not recognise how to link the sum to infinity to the total distance travelled.

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The three planes having Cartesian equations $2x + 3y - z = 11,\ x + 2y + z = 3$ and $5x - y - z = 10$ meet at a point P. Find the coordinates of P

.

Markscheme

using technology and/or by elimination (*eg* ref on GDC) ***(M1)***

$x = 1.89\ \left(= \frac{17}{9}\right),\ y = 1.67\ \left(= \frac{5}{3}\right),\ z = -2.22\ \left(= \frac{-20}{9}\right)$ ***A1A1A1***

Note: Award ***A1A1A0*** for a set of correct answers not given exactly or to three significant figures.

[4 marks]

Examiners report

[N/A]

(a) Show that the complex number i is a root of the equation

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0\ .$$

(b) Find the other roots of this equation.

Markscheme

(a) $\mathrm{i}^4 - 5\mathrm{i}^3 + 7\mathrm{i}^2 - 5\mathrm{i} + 6 = 1 + 5\mathrm{i} - 7 - 5\mathrm{i} + 6$ ***M1A1***

$= 0$ ***AG*** ***N0***

(b) i root $\Rightarrow -\mathrm{i}$ is second root ***(M1)A1***

moreover, $x^4 - 5x^3 + 7x^2 - 5x + 6 = (x - \mathrm{i})\,(x + \mathrm{i})\,q(x)$

where $q(x) = x^2 - 5x + 6$

finding roots of $q(x)$

the other two roots are 2 and 3 ***A1A1***

Note: Final ***A1A1*** is independent of previous work.

[6 marks]

Examiners report

A surprising number of candidates solved the question by dividing the expression by $1-i$ rather than substituting l into the expression. Many students were not aware that complex roots occur in conjugate pairs, and many did not appreciate the difference between a factor and a root. Generally the question was well done.

From a group of five males and six females, four people are chosen.

a. Determine how many possible groups can be chosen. [2]

b. Determine how many groups can be formed consisting of two males and two females. [2]

c. Determine how many groups can be formed consisting of at least one female. [2]

Markscheme

a. $\binom{11}{4} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$ ***(M1)A1***

[2 marks]

b.

$\binom{5}{2} \times \binom{6}{2} = \frac{5\times4}{2\times1} \times \frac{6\times5}{2\times1}$ ***M1***

$= 150$ ***A1***

[2 marks]

c. **METHOD 1**

number of ways all men $= \binom{5}{4} = 5$

$330 - 5 = 325$ ***M1A1***

Note: Allow FT from answer obtained in part (a).

[2 marks]

METHOD 2

$\binom{6}{1}\binom{5}{3} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{1} + \binom{6}{4}$ ***M1***

$= 325$ ***A1***

[2 marks]

Total [6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Consider the following system of equations

$$2x + y + 6z = 0$$
$$4x + 3y + 14z = 4$$
$$2x - 2y + (\alpha - 2)z = \beta - 12.$$

a. Find conditions on α and β for which [6]

(i) the system has no solutions;

(ii) the system has only one solution;

(iii) the system has an infinite number of solutions.

b. In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. [3]

Markscheme

a. $2x + y + 6z = 0$

$4x + 3y + 14z = 4$

$2x - 2y + (\alpha - 2)z = \beta - 12$

attempt at row reduction ***M1***

eg $R_2 - 2R_1$ and $R_3 - R_1$

$2x + y + 6z = 0$

$y + 2z = 4$

$-3y + (\alpha - 8)z = \beta - 12$ ***A1***

eg $R_3 + 3R_2$

$2x + y + 6z = 0$

$y + 2z = 4$ ***A1***

$(\alpha - 2)z = \beta$

(i) no solutions if $\alpha = 2,\ \beta \ne 0$ ***A1***

(ii) one solution if $\alpha \ne 2$ ***A1***

(iii) infinite solutions if $\alpha = 2,\ \beta = 0$ ***A1***

Note: Accept alternative methods e.g. determinant of a matrix

Note: Award ***A1A1A0*** if all three consistent with their reduced form, ***A1A0A0*** if two or one answer consistent with their reduced form.

[6 marks]

b. $y + 2z = 4 \Rightarrow y = 4 - 2z$

$2x = -y - 6z = 2z - 4 - 6z = -4z - 4 \Rightarrow x = -2z - 2$ ***A1***

therefore Cartesian equation is $\frac{x+2}{-2} = \frac{y-4}{-2} = \frac{z}{1}$ or equivalent ***A1***

[3 marks]

Total [9 marks]

Examiners report

a. [N/A]
b. [N/A]

One root of the equation $x^2 + ax + b = 0$ is $2 + 3\text{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b.

Markscheme

METHOD 1

substituting
$-5 + 12\text{i} + a(2 + 3\text{i}) + b = 0$ ***(A1)***
equating real or imaginary parts ***(M1)***
$12 + 3a = 0 \Rightarrow a = -4$ ***A1***
$-5 + 2a + b = 0 \Rightarrow b = 13$ ***A1***
METHOD 2
other root is $2 - 3\text{i}$ ***(A1)***
considering either the sum or product of roots or multiplying factors ***(M1)***
$4 = -a$ (sum of roots) so $a = -4$ ***A1***
$13 = b$ (product of roots) ***A1***
[4 marks]

Examiners report

[N/A]

---

a. Find the set of values of x for which the series $\sum_{n=1}^{\infty}\left(\frac{2x}{x+1}\right)^n$ has a finite sum. [4]

b. Hence find the sum in terms of x. [2]

Markscheme

a. for the series to have a finite sum, $\left|\frac{2x}{x+1}\right| < 1$ ***R1***

(sketch from gdc or algebraic method) ***M1***

S_∞ exists when $-\frac{1}{3} < x < 1$ ***A1A1***

Note: Award ***A1*** for bounds and ***A1*** for strict inequalities.

[4 marks]

b. $S_\infty = \frac{\frac{2x}{x+1}}{1-\frac{2x}{x+1}} = \frac{2x}{1-x}$ ***M1A1***

[2 marks]

Examiners report

a. A large number of candidates omitted the absolute value sign in the inequality in (a), or the use of the correct double inequality. Among candidates who had the correct statement, those who used their GDC were the most successful. The algebraic solution of the inequality was difficult for some candidates. In (b), quite a number of candidates found the sum of the first *n* terms of the geometric series, rather than the infinite sum of the series.

b. A large number of candidates omitted the absolute value sign in the inequality in (a), or the use of the correct double inequality. Among candidates who had the correct statement, those who used their GDC were the most successful. The algebraic solution of the inequality was difficult for some candidates. In (b), quite a number of candidates found the sum of the first *n* terms of the geometric series, rather than the infinite sum of the series.

The system of equations

$$2x - y + 3z = 2$$

$$3x + y + 2z = -2$$

$$-x + 2y + az = b$$

is known to have more than one solution. Find the value of a and the value of b.

Markscheme

EITHER

using row reduction (or attempting to eliminate a variable) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 3 & 1 & 2 & -2 \\ -1 & 2 & a & b \end{array}\right) \begin{array}{l} \\ \to 2R2 - 3R1 \\ \to 2R3 + R1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & -10 \\ 0 & 3 & 2a+3 & 2b+2 \end{array}\right) \to R2/5$$ ***A1***

Note: For an algebraic solution award ***A1*** for **two** correct equations in two variables.

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & 2a+3 & 2b+2 \end{array}\right) \to R3 - 3R2$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2a+6 & 2b+8 \end{array}\right)$$

Note: Accept alternative correct row reductions.

recognition of the need for 4 zeroes ***M1***

so for multiple solutions $a = -3$ and $b = -4$ ***A1A1***

[5 marks]

OR

$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & a \end{vmatrix} = 0$ ***M1***

$\Rightarrow 2(a-4) + (3a+2) + 3(6+1) = 0$

$\Rightarrow 5a + 15 = 0$

$\Rightarrow a = -3$ ***A1***

$\begin{vmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ -1 & 2 & b \end{vmatrix} = 0$ ***M1***

$\Rightarrow 2(b+4) + (3b-2) + 2(6+1) = 0$ ***A1***

$\Rightarrow 5b + 20 = 0$

$\Rightarrow b = -4$ ***A1***

[5 marks]

Examiners report

Many candidates attempted an algebraic approach that used excessive time but still allowed few to arrive at a solution. Of those that recognised the question should be done by matrices, some were unaware that for more than one solution a complete line of zeros is necessary.

Consider $w = \frac{z}{z^2+1}$ where $z = x + \mathrm{i}y$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that $\mathrm{Im}\, w = 0$, show that $|z| = 1$.

Markscheme

METHOD 1

Substituting $z = x + \mathrm{i}y$ to obtain $w = \frac{x+y\mathrm{i}}{(x+y\mathrm{i})^2+1}$ ***(A1)***

$w = \frac{x+y\mathrm{i}}{x^2-y^2+1+2xy\mathrm{i}}$ ***A1***

Use of $(x^2 - y^2 + 1 + 2xy\mathrm{i})$ to make the denominator real. ***M1***

$= \frac{(x+y\mathrm{i})(x^2-y^2+1-2xy\mathrm{i})}{(x^2-y^2+1)^2+4x^2y^2}$ ***A1***

$\mathrm{Im}\, w = \frac{y(x^2-y^2+1)-2x^2y}{(x^2-y^2+1)^2+4x^2y^2}$ **(A1)**

$= \frac{y(1-x^2-y^2)}{(x^2-y^2+1)^2+4x^2y^2}$ ***A1***

$\mathrm{Im}\, w = 0 \Rightarrow 1 - x^2 - y^2 = 0$ *i.e.* $|z| = 1$ as $y \neq 0$ ***R1AG*** ***N0***

[7 marks]

METHOD 2

$w(z^2+1) = z$ ***(A1)***

$w(x^2 - y^2 + 1 + 2\mathrm{i}xy) = x + yi$ ***A1***

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1,\ y \neq 0$ ***M1A1***

Substituting $w = \frac{1}{2x}$ to give $\frac{x}{2} - \frac{y^2}{2x} + \frac{1}{2x} = x$ ***A1***

$-\frac{1}{2x}(y^2 - 1) = \frac{x}{2}$ or equivalent ***(A1)***

$x^2 + y^2 = 1$, *i.e.* $|z| = 1$ as $y \neq 0$ ***R1AG***

[7 marks]

Examiners report

This was a difficult question that troubled most candidates. Most candidates were able to substitute $z = x + y\mathrm{i}$ into w but were then unable to make any further meaningful progress. Common errors included not expanding $(x + \mathrm{i}y)^2$ correctly or not using a correct complex conjugate to make the denominator real. A small number of candidates produced correct solutions by using $w = \frac{1}{z + z^{-1}}$.

The equations of three planes, are given by

$$ax + 2y + z = 3$$
$$-x + (a+1)\,y + 3z = 1$$
$$-2x + y + (a+2)\,z = k$$

where $a \in \mathbb{R}$.

a. Given that $a = 0$, show that the three planes intersect at a point. [3]

b. Find the value of a such that the three planes do not meet at a point. [5]

c. Given a such that the three planes do not meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form [6]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

Markscheme

a. $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}$

$$\begin{vmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{vmatrix} = 0 - 2(-2+6) + (-1+2) = -7 \quad \textit{M1A1}$$

since determinant $\neq 0 \Rightarrow$ unique solution to the system ***R1***

planes intersect in a point ***AG***

Note: For any method, including row reduction, leading to the explicit solution $\left(\frac{6-5k}{7}, \frac{10+k}{7}, \frac{1-2k}{7}\right)$, award ***M1*** for an attempt at a correct method, ***A1*** for two correct coordinates and ***A1*** for a third correct coordinate.

[3 marks]

b. $$\begin{vmatrix} a & 2 & 1 \\ -1 & a+1 & 3 \\ -2 & 1 & a+2 \end{vmatrix} = a\left((a+1)(a+2) - 3\right) - 2\left(-1(a+2) + 6\right) + \left(-1 + 2(a+1)\right) \quad \textit{M1(A1)}$$

planes not meeting in a point $\Rightarrow$ no unique solution *i.e.* determinant $= 0$ ***(M1)***

$a\left(a^2+3a-1\right) + (2a-8) + (2a+1) = 0$

$a^3+3a^2+3a-7=0$ ***A1***

\(a =1\) ***A1***

[5 marks]

c. $$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ -2 & 1 & 3 & k \end{pmatrix} \begin{matrix} r_1 + r_2 \\ \\ \\ \end{matrix} \qquad \textit{M1}$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 5 & 5 & 6+k \end{pmatrix} \begin{matrix} 2r_1 + r_3 \\ \\ \\ \end{matrix} \qquad \textit{(A1)}$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 4+4k \end{pmatrix} \begin{matrix} 4r_3 + 5r_2 \\ \\ \\ \end{matrix} \qquad \textit{(A1)}$$

for an infinite number of solutions to exist, $4+4k=0 \Rightarrow \ k=-1$ ***A1***

$x+2y+z=3$

$y+z=1$ ***M1***

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \textit{A1}$$

Note: Accept methods involving elimination.

Note: Accept any equivalent form *e.g.* $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Award ***A0*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$ or $r =$ is absent.

[6 marks]

Examiners report

a. It was disappointing to see that a significant number of candidates did not appear to be well prepared for this question and made no progress at all. There were a number of schools where no candidate made any appreciable progress with the question. This was of concern since this is quite a standard problem in Mathematics HL exams. Parts (a) and (b) were intended to be answered by the use of determinants, but many candidates were not aware of this technique and used elimination. Whilst a valid method, elimination led to a long and cumbersome solution when a much more straightforward solution was available using determinants.

b. It was disappointing to see that a significant number of candidates did not appear to be well prepared for this question and made no progress at all. There were a number of schools where no candidate made any appreciable progress with the question. This was of concern since this is quite a standard problem in Mathematics HL exams. Parts (a) and (b) were intended to be answered by the use of determinants, but many candidates were not aware of this technique and used elimination. Whilst a valid method, elimination led to a long and cumbersome solution when a much more straightforward solution was available using determinants.

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a. (i) Express the sum of the first *n* positive odd integers using sigma notation. [4]

(ii) Show that the sum stated above is n^2.

(iii) Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

b. A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points. [7]

(i) Show on a diagram all diagonals if there are 5 points.

(ii) Show that the number of diagonals is $\frac{n(n-3)}{2}$ if there are n points, where $n > 2$.

(iii) Given that there are more than one million diagonals, determine the least number of points for which this is possible.

c. The random variable $X \sim B(n,\ p)$ has mean 4 and variance 3. [8]

(i) Determine *n* and *p*.

(ii) Find the probability that in a single experiment the outcome is 1 or 3.

Markscheme

a. (i) $\sum\limits_{k=1}^{n} (2k-1)$ (or equivalent) ***A1***

Note: Award ***A0*** for $\sum\limits_{n=1}^{n} (2n-1)$ or equivalent.

(ii) **EITHER**

$2 \times \frac{n(n+1)}{2} - n$ ***M1A1***

OR

$\frac{n}{2}(2 + (n-1)2)$ (using $S_n = \frac{n}{2}(2u_1 + (n-1)d)$) ***M1A1***

OR

$\frac{n}{2}(1 + 2n - 1)$ (using $S_n = \frac{n}{2}(u_1 + u_n)$) ***M1A1***

THEN

$= n^2$ ***AG***

(iii) $47^2 - 14^2 = 2013$ ***A1***

[4 marks]

b. (i) **EITHER**

a pentagon and five diagonals ***A1***

OR

five diagonals (circle optional) ***A1***

(ii) Each point joins to $n - 3$ other points. ***A1***

a correct argument for $n(n-3)$ ***R1***

a correct argument for $\frac{n(n-3)}{2}$ ***R1***

(iii) attempting to solve $\frac{1}{2}n(n-3) > 1\,000\,000$ for n. ***(M1)***

$n > 1415.7$ ***(A1)***

$n = 1416$ ***A1***

[7 marks]

c. (i) $np = 4$ and $npq = 3$ ***(A1)***

attempting to solve for n and p ***(M1)***

$n = 16$ and $p = \frac{1}{4}$ ***A1***

(ii) $X \sim B(16, 0.25)$ ***(A1)***

$P(X = 1) = 0.0534538...(= \binom{16}{1}(0.25)(0.75)^{15})$ ***(A1)***

$P(X = 3) = 0.207876...(= \binom{16}{3}(0.25)^3(0.75)^{13})$ ***(A1)***

$\mathrm{P}(X = 1) + \mathrm{P}(X = 3)$ ***(M1)***

$= 0.261$ ***A1***

[8 marks]

Examiners report

a. In part (a) (i), a large number of candidates were unable to correctly use sigma notation to express the sum of the first *n* positive odd integers. Common errors included summing $2n - 1$ from 1 to *n* and specifying sums with incorrect limits. Parts (a) (ii) and (iii) were generally well done.

b. Parts (b) (i) and (iii) were generally well done. In part (b) (iii), many candidates unnecessarily simplified their quadratic when direct GDC use could have been employed. A few candidates gave $n > 1416$ as their final answer. While some candidates displayed sound reasoning in part (b) (ii), many candidates unfortunately adopted a 'proof by example' approach.

c. Part (c) was generally well done. In part (c) (ii), some candidates multiplied the two probabilities rather than adding the two probabilities.

Given that $z = \frac{2-\mathrm{i}}{1+\mathrm{i}} - \frac{6+8\mathrm{i}}{u+\mathrm{i}}$, find the values of u, $u \in \mathbb{R}$, such that $\operatorname{Re} z = \operatorname{Im} z$.

Markscheme

METHOD 1

$\frac{2-\mathrm{i}}{1+\mathrm{i}} = \frac{1-3\mathrm{i}}{2}$ ***A1***

$\frac{6+8\mathrm{i}}{u+\mathrm{i}} \times \frac{u-\mathrm{i}}{u-\mathrm{i}} = \frac{6u+8+(8u-6)\mathrm{i}}{u^2+1}$ ***M1A1***

$\Rightarrow \frac{2-\mathrm{i}}{1+\mathrm{i}} - \frac{6+8u}{u+\mathrm{i}} = \frac{1}{2} - \frac{6u+8}{u^2+1} - \left(\frac{3}{2} + \frac{8u-6}{u^2+1}\right)\mathrm{i}$

$\operatorname{Im} z = \operatorname{Re} z$

$\Rightarrow \frac{1}{2} - \frac{6u+8}{u^2+1} = -\frac{3}{2} - \frac{8u-6}{u^2+1}$ ***A1***

(sketch from gdc, or algebraic method) ***(M1)***

$u = -3$; $u = 2$ ***A1A1*** ***N2***

[7 marks]

METHOD 2

$\frac{2-\mathrm{i}}{1+\mathrm{i}} - \frac{6+8\mathrm{i}}{u+\mathrm{i}} = \frac{(2-\mathrm{i})(u+\mathrm{i})-(1+\mathrm{i})(6+8\mathrm{i})}{(u-1)+\mathrm{i}(u+1)}$ ***M1A1***

$= \frac{(2-\mathrm{i})(u+\mathrm{i})-(1+\mathrm{i})(6+8\mathrm{i})}{(u-1)+\mathrm{i}(u+1)} \cdot \frac{(u-1)-\mathrm{i}(u+1)}{(u-1)-\mathrm{i}(u+1)}$ ***M1***

$= \frac{u^2-12u-15+\mathrm{i}(-3u^2-16u+9)}{2(u^2+1)}$ ***A1***

$\operatorname{Re} z = \operatorname{Im} z \Rightarrow u^2 - 12u - 15 = -3u^2 - 16u + 9$ ***M1***

$u = -3$; $u = 2$ ***A1A1*** ***N2***

[7 marks]

Examiners report

Many candidates failed to access their GDC early enough to avoid huge algebraic manipulations, often carried out with many errors. Some candidates failed to separate and equate the real and imaginary parts of the expression obtained.

The fourth term in an arithmetic sequence is 34 and the tenth term is 76.

(a) Find the first term and the common difference.

(b) The sum of the first *n* terms exceeds 5000. Find the least possible value of *n*.

Markscheme

(a) **METHOD 1**

$34 = a + 3d$ and $76 = a + 9d$ ***(M1)***

$d = 7$ ***A1***

$a = 13$ ***A1***

METHOD 2

$76 = 34 + 6d$ ***(M1)***

$d = 7$ ***A1***

$34 = a + 21$

$a = 13$ ***A1***

[3 marks]

(b) $\frac{n}{2}(26 + 7(n-1)) > 5000$ ***(M1)(A1)***

$n > 36.463\ldots$ ***(A1)***

Note: Award ***M1A1A1*** for using either an equation, a graphical approach or a numerical approach.

$n = 37$ ***A1*** ***N3***

[4 marks]

Total [7 marks]

Examiners report

Both parts were very well done. In part (a), a few candidates made a careless algebraic error when attempting to find the value of *a* or *d*.

In part (b), a few candidates attempted to find the value of *n* for which $u_n > 5000$. Some candidates used the incorrect formula $S_n = \frac{n}{2}[u_1 + (n-1)d]$. A number of candidates unnecessarily attempted to simplify S_n. Most successful candidates in part (b) adopted a graphical approach and communicated their solution effectively. A few candidates did not state their value of *n* as an integer.

It is known that the number of fish in a given lake will decrease by 7% each year unless some new fish are added. At the end of each year, 250 new fish are added to the lake.

At the start of 2018, there are 2500 fish in the lake.

a. Show that there will be approximately 2645 fish in the lake at the start of 2020. [3]

b. Find the approximate number of fish in the lake at the start of 2042. [5]

Markscheme

a. **EITHER**

2019: 2500 × 0.93 + 250 = 2575 ***(M1)A1***

2020: 2575 × 0.93 + 250 ***M1***

OR

2020: 2500 × 0.93^2 + 250(0.93 + 1) ***M1M1A1***

Note: Award ***M1*** for starting with 2500, ***M1*** for multiplying by 0.93 and adding 250 twice. ***A1*** for correct expression. Can be shown in recursive form.

THEN

(= 2644.75) = 2645 ***AG***

[3 marks]

b. 2020: 2500 × 0.93^2 + 250(0.93 + 1)

2042: 2500 × 0.93^{24} + 250(0.93^{23} + 0.93^{22} + … + 1) ***(M1)(A1)***

$= 2500 \times 0.93^{24} + 250\frac{(0.93^{24}-1)}{(0.93-1)}$ ***(M1)(A1)***

=3384 ***A1***

Note: If recursive formula used, award ***M1*** for $u_n = 0.93\, u_{n-1}$ **and** u_0 or u_1 seen (can be awarded if seen in part (a)). Then award ***M1A1*** for attempt to find u_{24} or u_{25} respectively (different term if other than 2500 used) (***M1A0*** if incorrect term is being found) and ***A2*** for correct answer.

Note: Accept all answers that round to 3380.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]

It has been suggested that in rowing competitions the time, T seconds taken to complete a 2000 m race can be modelled by an equation of the form $T = aN^b$, where N is the number of rowers in the boat and a and b are constants for rowers of a similar standard.

To test this model the times for the finalists in all the 2000 m men's races at a recent Olympic games were recorded and the mean calculated.

The results are shown in the following table for $N = 1$ and $N = 2$.

N	T (seconds)
1	420.65
2	390.94

It is now given that the mean time in the final for boats with 8 rowers was 342.08 seconds.

a. Use these results to find estimates for the value of a and the value of b. Give your answers to five significant figures. [4]

b. Use this model to estimate the mean time for the finalists in an Olympic race for boats with 8 rowers. Give your answer correct to two decimal places. [1]

c. Calculate the error in your estimate as a percentage of the actual value. [1]

d. Comment on the likely validity of the model as N increases beyond 8. [2]

Markscheme

a. $a = 420.65$ ***A1***

$390.94 = a \times 2^b$ ***M1***

$2^b = \frac{390.94}{420.65} = 0.929\ldots$ ***A1***

$b = -0.10567$ ***A1***

[4 marks]

b. $N = 8$ $T = 337.67$ ***A1***

Note: Accept 5sf answers between 337.44 and 337.67.

[1 mark]

c. $N = 8$ Percentage error 1.29% ***A1***

Note: Accept negative values of the above.

[1 mark]

d. likely not to be a good fit for larger values of N ***R1***

likely to be quite a good fit for values close to 8 ***R1***

[2 marks]

Examiners report

a. Parts (a) to (c) were generally well done, although far too much inaccuracy with basic calculations.

b. Parts (a) to (c) were generally well done, although far too much inaccuracy with basic calculations.

c. Parts (a) to (c) were generally well done, although far too much inaccuracy with basic calculations.

d. Parts (a) to (c) were generally well done, although far too much inaccuracy with basic calculations. Part (d) caused more difficulties as candidates frequently had insufficient analysis to gain the two marks.

The three planes

$2x - 2y - z = 3$

$4x + 5y - 2z = -3$

$3x + 4y - 3z = -7$

intersect at the point with coordinates (a, b, c).

a. Find the value of each of a, b and c. [2]

b. The equations of three planes are [4]

$2x - 4y - 3z = 4$

$-x + 3y + 5z = -2$

$3x - 5y - z = 6.$

Find a vector equation of the line of intersection of these three planes.

Markscheme

a. (a) use GDC or manual method to find a, b and c ***(M1)***

obtain $a = 2,\ b = -1,\ c = 3$ (in any identifiable form) ***A1***

[2 marks]

b. use GDC or manual method to solve second set of equations ***(M1)***

obtain $x = \frac{4-11t}{2};\ y = \frac{-7t}{2};\ z = t$ (or equivalent) ***(A1)***

$r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5.5 \\ -3.5 \\ 1 \end{pmatrix}$ (accept equivalent vector forms) ***M1A1***

Note: Final ***A1*** requires *r* = or equivalent.

[4 marks]

Examiners report

a. Generally well done.

b. Moderate success here. Some forgot that an equation must have an = sign.

Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0,\ n \in \mathbb{Z}^+;$

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0,\ z \neq -1.$

Markscheme

(a) using de Moivre's theorem

$z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta\ (= 2\cos n\theta)$, imaginary part of which is 0 ***M1A1***

so $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0$ ***AG***

(b) $\frac{z-1}{z+1} = \frac{\cos\theta + \mathrm{i}\sin\theta - 1}{\cos\theta + \mathrm{i}\sin\theta + 1}$

$= \frac{(\cos\theta - 1 + \mathrm{i}\sin\theta)(\cos\theta + 1 - \mathrm{i}\sin\theta)}{(\cos\theta + 1 + \mathrm{i}\sin\theta)(\cos\theta + 1 - \mathrm{i}\sin\theta)}$ ***M1A1***

Note: Award ***M1*** for an attempt to multiply numerator and denominator by the complex conjugate of their denominator.

$\Rightarrow \mathrm{Re}\left(\frac{z-1}{z+1}\right) = \frac{(\cos\theta-1)(\cos\theta+1)+\sin^2\theta}{\text{real denominator}}$ ***M1A1***

Note: Award ***M1*** for multiplying out the numerator.

$= \frac{\cos^2\theta+\sin^2\theta-1}{\text{real denominator}}$ ***A1***

$= 0$ ***AG***

[7 marks]

Examiners report

Part(a) - The majority either obtained full marks or no marks here.

Part(b) - This question was algebraically complex and caused some candidates to waste their efforts for little credit.

---

a. Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. [3]

b. Hence find the least value of n for which $\binom{3n+1}{3n-2} > 10^6$. [3]

Markscheme

a. $\binom{3n+1}{3n-2} = \frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$= \frac{(3n+1)3n(3n-1)}{3!}$ ***A1***

$= \frac{9}{2}n^3 - \frac{1}{2}n$ or equivalent ***A1***

[3 marks]

b. attempt to solve $= \frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

Examiners report

a. [N/A]

b. [N/A]

Three Mathematics books, five English books, four Science books and a dictionary are to be placed on a student's shelf so that the books of each subject remain together.

(a) In how many different ways can the books be arranged?

(b) In how many of these will the dictionary be next to the Mathematics books?

Markscheme

(a) There are 3! ways of arranging the Mathematics books, 5! ways of arranging the English books and 4! ways of arranging the Science books. ***(A1)***

Then we have 4 types of books which can be arranged in 4! ways. ***(A1)***

$3! \times 5! \times 4 \times 4! = 414\,720$ ***(M1)A1***

(b) There are 3! ways of arranging the subject books, and for each of these there are 2 ways of putting the dictionary next to the Mathematics books. ***(M1)(A1)***

$3! \times 5! \times 4! \times 3! \times 2 = 207\,360$ ***A1***

[7 marks]

Examiners report

Many students added instead of multiplying. There were, however, quite a few good answers to this question.

Twelve students are to take an exam in advanced combinatorics.

The exam room is set out in three rows of four desks, with the invigilator at the front of the room, as shown in the following diagram.

INVIGILATOR

Desk 1	Desk 2	Desk 3	Desk 4
Desk 5	Desk 6	Desk 7	Desk 8
Desk 9	Desk 10	Desk 11	Desk 12

Two of the students, Helen and Nicky, are suspected of cheating in a previous exam.

a. Find the number of ways the twelve students may be arranged in the exam hall. [1]

b. Find the number of ways the students may be arranged if Helen and Nicky must sit so that one is directly behind the other (with no desk in between). For example Desk 5 and Desk 9. [2]

c. Find the number of ways the students may be arranged if Helen and Nicky must not sit next to each other in the same row. [3]

Markscheme

a. $12!\ (= 479001600)$ ***A1***

[1 mark]

b. **METHOD 1**

$8 \times 2 = 16$ ways of sitting Helen and Nicky, 10! ways of sitting everyone else ***(A1)***

$16 \times 10!$

$= 58060800$ ***A1***

METHOD 2

$8 \times 1 \times 10!\ (= 29030400)$ ways if Helen sits in the front or back row

$4 \times 2 \times 10!\ (= 29030400)$ ways if Helen sits in the middle row ***(A1)***

Note: Award ***A1*** for one correct value.

2×29030400

$= 58060800$ ***A1***

[2 marks]

c. **METHOD 1**

$9 \times 2 \times 0!\ (= 65318400)$ ways if Helen and Nicky sit next to each other ***(A1)***

attempt to subtract from total number of ways ***(M1)***

$12! - 9 \times 2 \times 10!$

$= 413683200$ ***A1***

METHOD 2

$6 \times 10 \times 10!\ (= 217728000)$ ways if Helen sits in column 1 or 4 ***(A1)***

$6 \times 9 \times 10!\ (= 195955200)$ ways if Helen sits in column 2 or 3 ***(A1)***

$217728000 + 195955200$

$= 413683200$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Three boys and three girls are to sit on a bench for a photograph.

a. Find the number of ways this can be done if the three girls must sit together. [3]

b. Find the number of ways this can be done if the three girls must all sit apart. [4]

Markscheme

a. the three girls can sit together in $3! = 6$ ways *(A1)*

this leaves 4 'objects' to arrange so the number of ways this can be done is 4! *(M1)*

so the number of arrangements is $6 \times 4! = 144$ ***A1***

[3 marks]

b. Finding more than one position that the girls can sit *(M1)*

Counting exactly four positions *(A1)*

number of ways $= 4 \times 3! \times 3! = 144$ ***M1A1*** ***N2***

[4 marks]

Examiners report

a. Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

b. Some good solutions to part (a) and certainly fewer completely correct answers to part (b). Many candidates were able to access at least partial credit, if they were showing their reasoning.

Prove, by mathematical induction, that $7^{8n+3} + 2,\ n \in \mathbb{N}$, is divisible by 5.

Markscheme

if $n = 0$

$7^3 + 2 = 345$ which is divisible by 5, hence true for $n = 0$ ***A1***

Note: Award ***A0*** for using $n = 1$ but do not penalize further in question.

assume true for $n = k$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

so $7^{8k+3} + 2 = 5p,\ p \in \bullet$ ***A1***
if $n = k + 1$
$7^{8(k+1)+3} + 2$ ***M1***
$= 7^8 7^{8k+3} + 2$ ***M1***
$= 7^8(5p - 2) + 2$ ***A1***
$= 7^8.5p - 2.7^8 + 2$
$= 7^8.5p - 11\,529\,600$
$= 5(7^8 p - 2\,305\,920)$ ***A1***
hence if true for $n = k$, then also true for $n = k + 1$. Since true for $n = 0$, then true for all $n \in \bullet$ ***R1***

Note: Only award the ***R1*** if the first two ***M1***s have been awarded.

[8 marks]

Examiners report

[N/A]

Consider the arithmetic sequence 8, 26, 44, $\ldots$.

(a) Find an expression for the n^{th} term.

(b) Write down the sum of the first *n* terms using sigma notation.

(c) Calculate the sum of the first 15 terms.

Markscheme

(a) $18n - 10$ (or equivalent) ***A1***

(b) $\sum_{1}^{n}(18r - 10)$ (or equivalent) ***A1***

(c) by use of GDC or algebraic summation or sum of an AP ***(M1)***

$\sum_{1}^{15}(18r - 10) = 2010$ ***A1***

[4 marks]

Examiners report

An easy starter question, but few candidates seem to be familiar with the conventions of sigma notation.

Find the sum of all three-digit natural numbers that are not exactly divisible by 3.

Markscheme

$(100 + 101 + 102 + \ldots + 999) - (102 + 105 + \ldots + 999)$ ***(M1)***

$= \frac{900}{2}(100 + 999) - \frac{300}{2}(102 + 999)$ ***M1A1A1***

$= 329\,400$ ***A1 N5***

Note: A variety of other acceptable methods may be seen including for example $\frac{300}{2}(201 + 1995)$ or $\frac{600}{2}(100 + 998)$.

[5 marks]

Examiners report

There were many good solutions seen by a variety of different methods.

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.

The account paid a fixed rate of 0.4% interest per month. The interest was calculated on the last day of each month and added to the account. Let $\$A_n$ be the amount in Mary's account on the last day of the $n\text{th}$ month, immediately after the interest had been added.

a. Find an expression for A_1 and show that $A_2 = 1.004^2x + 1.004x$. [2]

b. (i) Write down a similar expression for A_3 and A_4. [6]

(ii) Hence show that the amount in Mary's account the day before she turned 10 years old is given by $251(1.004^{120} - 1)x$.

c. Write down an expression for A_n in terms of x on the day before Mary turned 18 years old showing clearly the value of n. [1]

d. Mary's grandparents wished for the amount in her account to be at least $\$20\,000$ the day before she was 18. Determine the minimum value of the monthly deposit $\$x$ required to achieve this. Give your answer correct to the nearest dollar. [4]

e. As soon as Mary was 18 she decided to invest $\$15\,000$ of this money in an account of the same type earning 0.4% interest per month. She withdraws $\$1000$ every year on her birthday to buy herself a present. Determine how long it will take until there is no money in the account. [5]

Markscheme

a. $A_1 = 1.004x$ ***A1***

$A_2 = 1.004(1.004x + x)$ ***A1***

$= 1.004^2x + 1.004x$ ***AG***

Note: Accept an argument in words for example, first deposit has been in for two months and second deposit has been in for one month.

[2 marks]

b. (i) $A_3 = 1.004(1.004^2x + 1.004x + x) = 1.004^3x + 1.004^2x + 1.004x$ ***(M1)A1***

$A_4 = 1.004^4x + 1.004^3x + 1.004^2x + 1.004x$ ***A1***

(ii) $A_{120} = (1.004^{120} + 1.004^{119} + \ldots + 1.004)x$ ***(A1)***

$= \frac{1.004^{120}-1}{1.004-1} \times 1.004x$ ***M1A1***

$= 251(1.004^{120} - 1)x$ ***AG***

[6 marks]

c. $A_{216} = 251(1.004^{216} - 1)x \left(= x\sum_{t=1}^{216} 1.004^t\right)$ ***A1***

[1 mark]

d. $251(1.004^{216} - 1)x = 20\,000 \Rightarrow x = 58.22\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for $251(1.004^{216} - 1)x > 20\,000$, ***(M1)*** for attempting to solve and ***(A1)*** for $x > 58.22\ldots$.

$x = 59$ ***A1***

Note: Accept $x = 58$. Accept $x \geqslant 59$.

[4 marks]

e. $r = 1.004^{12}\ (= 1.049\ldots)$ ***(M1)***

$15\,000r^n - 1000\frac{r^n-1}{r-1} = 0 \Rightarrow n = 27.8\ldots$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for the equation (with their value of r), ***(M1)*** for attempting to solve for n and ***(A1)*** for $n = 27.8\ldots$

$n = 28$ ***A1***

Note: Accept $n = 27$.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

The complex numbers u and v are represented by point A and point B respectively on an Argand diagram.

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A'. Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B'.

a. Consider $z = r(\cos\theta + \mathrm{i}\sin\theta),\ z \in \mathbb{C}$. [7]

Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta),\ n \in \mathbb{Z}^+$.

b. Given $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 - \mathrm{i}$, [4]

(i) express u and v in modulus-argument form;

(ii) hence find u^3v^4.

c. Plot point A and point B on the Argand diagram. [1]

d. Find the area of triangle $\mathrm{OA'B'}$. [3]

e. Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b,\ c,\ d,\ e \in \mathbb{R}$, [5]

find the values of $b,\ c,\ d$ and e.

Markscheme

a. let $\mathrm{P}(n)$ be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), n \in ¢^+$

let $n = 1 \Rightarrow$
$\mathrm{LHS} = r(\cos\theta + \mathrm{i}\sin\theta)$
$\mathrm{RHS} = r(\cos\theta + \mathrm{i}\sin\theta),\ \therefore \mathrm{P}(1)$ is true ***R1***
assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

now show $n = k$ true implies $n = k+1$ also true
$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***M1***
$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$
$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$ ***A1***
$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$ ***A1***
$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k+1$ is true ***A1***
$\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true and $\mathrm{P}(1)$ is true, therefore by mathematical induction statement is true for $n \geqslant 1$ ***R1***

Note: Only award the final ***R1*** if the first 4 marks have been awarded.

[7 marks]

b. (i) $u = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$ ***A1***
$v = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$ ***A1***

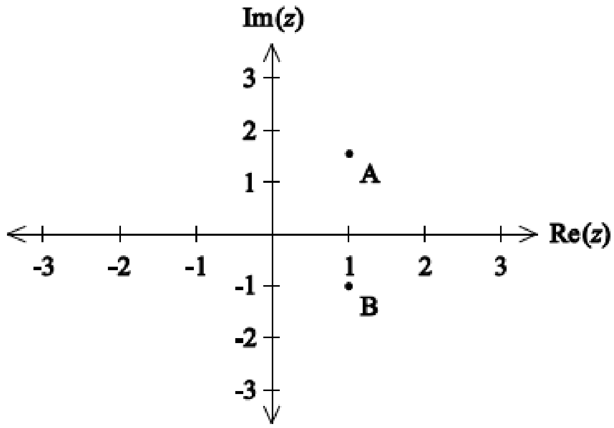
Notes: Accept 3 sf answers only. Accept equivalent forms.
Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\mathrm{cis}(\pi) = -8$
$v^4 = 4\mathrm{cis}(-\pi) = -4$ ***(M1)***
$u^3v^4 = 32$ ***A1***

Notes: Award ***(M1)*** for an attempt to find u^3 and v^4.
Accept equivalent forms.

[4 marks]

c.



A1

Note: Award **A1** if A or $1+\sqrt{3}i$ and B or $1-i$ are in their correct quadrants, are aligned vertically and it is clear that $|u|>|v|$.

[1 mark]

d. Area $=\frac{1}{2}\times 2\times\sqrt{2}\times\sin\left(\frac{5\pi}{12}\right)$ ***M1A1***

$=1.37\ \left(=\frac{\sqrt{2}}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)$ ***A1***

Notes: Award ***M1A0A0*** for using $\frac{7\pi}{12}$.

[3 marks]

e. $(z-1+\mathrm{i})(z-1-\mathrm{i})=z^2-2z+2$ ***M1A1***

Note: Award ***M1*** for recognition that a complex conjugate is also a root.

$\left(z-1-\sqrt{3}\mathrm{i}\right)\left(z-1+\sqrt{3}\mathrm{i}\right)=z^2-2z+4$ ***A1***

$\left(z^2-2z+2\right)\left(z^2-2z+4\right)=z^4-4z^3+10z^2-12z+8$ ***M1A1***

Note: Award ***M1*** for an attempt to expand two quadratics.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

Phil takes out a bank loan of $150 000 to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and added to the amount outstanding.

To pay off the loan, Phil makes annual deposits of $*P* at the end of every year in a savings account, paying an annual interest rate of 2% . He makes his first deposit at the end of the first year after taking out the loan.

David visits a different bank and makes a single deposit of $*Q* , the annual interest rate being 2.8%.

a. Find the amount Phil would owe the bank after 20 years. Give your answer to the nearest dollar. [3]

b. Show that the total value of Phil's savings after 20 years is $\frac{(1.02^{20}-1)P}{(1.02-1)}$. [3]

c. Given that Phil's aim is to own the house after 20 years, find the value for P to the nearest dollar. [3]

d.i. David wishes to withdraw $5000 at the end of each year for a period of n years. Show that an expression for the minimum value of Q is [3]

$\frac{5000}{1.028}+\frac{5000}{1.028^2}+\ldots+\frac{5000}{1.028^n}$.

d.ii. Hence or otherwise, find the minimum value of Q that would permit David to withdraw annual amounts of $5000 indefinitely. Give your answer to the nearest dollar. [3]

Markscheme

a. 150000×1.035^{20} ***(M1)(A1)***

$=\$298468$ ***A1***

Note: Only accept answers to the nearest dollar. Accept $298469.

[3 marks]

b. attempt to look for a pattern by considering 1 year, 2 years *etc* ***(M1)***

recognising a geometric series with first term P and common ratio 1.02 ***(M1)***

EITHER

$P+1.02P+\ldots+1.02^{19}P\ \left(=P(1+1.02+\ldots+1.02^{19})\right)$ ***A1***

OR

explicitly identify $u_1=P,\ r=1.02$ and $n=20$ (may be seen as S_{20}). ***A1***

THEN

$s_{20}=\frac{(1.02^{20}-1)P}{(1.02-1)}$ ***AG***

[3 marks]

c. $24.297\ldots P=298468$ ***(M1)(A1)***

$P=12284$ ***A1***

Note: Accept answers which round to 12284.

[3 marks]

d.i. **METHOD 1**

$Q(1.028^n) = 5000(1 + 1.028 + 1.028^2 + 1.028^3 + \ldots + 1.028^{n-1})$ ***M1A1***

$Q = \frac{5000(1+1.028+1.028^2+1.028^3+\ldots+1.028^{n-1})}{1.028^n}$ ***A1***

$= \frac{5000}{1.028} + \frac{5000}{1.028^2} + \ldots + \frac{5000}{1.028^n}$ ***AG***

METHOD 2

the initial value of the first withdrawal is $\frac{5000}{1.028}$ ***A1***

the initial value of the second withdrawal is $\frac{5000}{1.028^2}$ ***R1***

the investment required for these two withdrawals is $\frac{5000}{1.028} + \frac{5000}{1.028^2}$ ***R1***

$Q = \frac{5000}{1.028} + \frac{5000}{1.028^2} + \ldots + \frac{5000}{1.028^n}$ ***AG***

[3 Marks]

d.ii. sum to infinity is $\frac{\frac{5000}{1.028}}{1-\frac{1}{1.028}}$ ***(M1)(A1)***

$= 178571.428\ldots$

so minimum amount is $178572 ***A1***

Note: Accept answers which round to $178571 or $178572.

[3 Marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d.i. [N/A]
d.ii. [N/A]

A system of equations is given below.

$x + 2y - z = 2$

$2x + y + z = 1$

$-x + 4y + az = 4$

(a) Find the value of a so that the system does not have a unique solution.

(b) Show that the system has a solution for any value of a.

Markscheme

(a) $\begin{bmatrix} x + 2y - z = 2 \\ 2x + y + z = 1 \\ -x + 4y + az = 4 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} x+2y-z=2 \\ -3y+3z=-3 \\ 6y+(a-1)z=6 \end{bmatrix} \quad \textit{M1A1}$$

$$\rightarrow \begin{bmatrix} x+2y-z=2 \\ -3y+3z=-3 \\ (a+5)z=0 \end{bmatrix} \quad \textit{A1}$$

(or equivalent)

if not a unique solution then $a = -5$ ***A1***

Note: The first ***M1*** is for attempting to eliminate a variable, the first ***A1*** for obtaining two expression in just two variables (plus a), and the second ***A1*** for obtaining an expression in just a and one other variable

[4 marks]

(b) if $a = -5$ there are an infinite number of solutions as last equation always true ***R1***

and if $a \neq -5$ there is a unique solution ***R1***

hence always a solution ***AG***

[2 marks]

Total [6 marks]

Examiners report

[N/A]

Let $\omega = \cos\theta + \mathrm{i}\sin\theta$. Find, in terms of θ , the modulus and argument of $(1-\omega^2)^*$.

Markscheme

METHOD 1

$(1-\omega^2)^* = (1-\mathrm{cis}\,2\theta)^* = ((1-\cos 2\theta) - \mathrm{i}\sin 2\theta)^*$ ***M1A1***

$= (1-\cos 2\theta) + \mathrm{i}\sin 2\theta$ ***A1***

$\left|(1-\omega^2)^*\right| = \sqrt{(1-\cos 2\theta)^2 + \sin^2 2\theta} \left(= \sqrt{(2\sin^2\theta)^2 + (2\sin\theta\cos\theta)^2} \right)$ ***M1***

$= |2\sin\theta|$ ***A1***

$\arg\left((1-\omega^2)^*\right) = \alpha \Rightarrow \tan\alpha = \cot(\theta)$ ***M1***

$\alpha = \frac{\pi}{2} - \theta$ ***A1***

therefore:

modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$

Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$

METHOD 2

EITHER

$(1-\varpi^2)^* = (1-\mathrm{cis}\,2\theta)^* = ((1-\cos 2\theta) - \mathrm{i}\sin 2\theta)^*$ ***M1A1***

$= (1 - \cos 2\theta) + \mathrm{i}\sin 2\theta$ ***A1***
$= (1 - 1 + 2\sin^2\theta) + 2\mathrm{i}\sin\theta\cos\theta$ ***M1***
OR
$(1 - \varpi^2)^* = \left(1 - (\cos\theta + \mathrm{i}\sin\theta)^2\right)^*$ ***M1A1***
$= (1 - \cos^2\theta + \sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta)^*$ ***A1***
$= 2\sin^2\theta + 2\mathrm{i}\sin\theta\cos\theta$ ***M1***
THEN
$= 2\sin\theta(\sin\theta + \mathrm{i}\cos\theta)$ ***(M1)***
$= 2\sin\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \theta\right)\right)$ ***A1A1***
$= 2\sin\theta\,\mathrm{cis}\left(\frac{\pi}{2} - \theta\right)$
therefore:
modulus is $2|\sin\theta|$ and argument is $\frac{\pi}{2} - \theta$ or $\frac{\pi}{2} - \theta \pm \pi$
Note: Accept modulus is $2\sin\theta$ and argument is $\frac{\pi}{2} - \theta$.

[7 marks]

Examiners report

This was the most challenging question in part A with just a few candidates scoring full marks. This question showed that many candidates have difficulties with algebraic manipulations, application of De Moivre's theorem and use of trigonometric identities. Although some candidates managed to calculate the square of a complex number, many failed to write down its conjugate or made algebraic errors which lead to wrong results in many cases. Just a few candidates were able to calculate the modulus and the argument of the complex number.

Fifteen boys and ten girls sit in a single line.

a. In how many ways can they be seated in a single line so that the boys and girls are in two separate groups? [3]

b. Two boys and three girls are selected to go the theatre. In how many ways can this selection be made? [3]

Markscheme

a. number of arrangements of boys is 15! and number of arrangements of girls is 10! ***(A1)***
total number of arrangements is $15! \times 10! \times 2 (= 9.49 \times 10^{18})$ ***M1A1***
Note: If 2 is omitted, award ***(A1)M1A0***.

[3 marks]

b. number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$ ***(A1)***
number of ways of choosing two boys and three girls is $\binom{15}{2} \times \binom{10}{3} = 12600$ ***M1A1***

[3 marks]

Examiners report

a. A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

b. A good number of correct answers were seen to this question, but a significant number of candidates forgot to multiply by 2 in part (a) and in part (b) the most common error was to add the combinations rather than multiply them.

$z_1 = (1 + \mathrm{i}\sqrt{3})^m$ and $z_2 = (1 - \mathrm{i})^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of *m* and *n*, respectively.

(b) **Hence**, find the smallest positive integers *m* and *n* such that $z_1 = z_2$.

Markscheme

(a) $\left|1 + \mathrm{i}\sqrt{3}\right| = 2$ or $\left|1 - \mathrm{i}\right| = \sqrt{2}$ ***(A1)***

$\arg(1 + \mathrm{i}\sqrt{3}) = \frac{\pi}{3}$ or $\arg(1 - \mathrm{i}) = -\frac{\pi}{4}$ $\left(\text{accept } \frac{7\pi}{4}\right)$ ***(A1)***

$|z_1| = 2^m$ ***A1***

$|z_2| = \sqrt{2}^n$ ***A1***

$\arg(z_1) = m \arctan\sqrt{3} = m\frac{\pi}{3}$ ***A1***

$\arg(z_2) = n \arctan(-1) = n\frac{-\pi}{4}$ $\left(\text{accept } n\frac{7\pi}{4}\right)$ ***A1*** ***N2***

[6 marks]

(b) $2^m = \sqrt{2}^n \Rightarrow n = 2m$ ***(M1)A1***

$m\frac{\pi}{3} = n\frac{-\pi}{4} + 2\pi k$, where *k* is an integer ***M1A1***

$\Rightarrow m\frac{\pi}{3} + n\frac{\pi}{4} = 2\pi k$

$\Rightarrow m\frac{\pi}{3} + 2m\frac{\pi}{4} = 2\pi k$ ***(M1)***

$\frac{5}{6}m\pi = 2\pi k$

$\Rightarrow m = \frac{12}{5}k$ ***A1***

The smallest value of *k* such that *m* is an integer is 5, hence

m =12 ***A1***

n = 24. ***A1*** ***N2***

[8 marks]

Total [14 marks]

Examiners report

Part (a) of this question was answered fairly well by candidates who attempted this question. The main error was the sign of the argument of z_2. Few candidates attempted part (b), and of those who did, most scored the first two marks for equating the modulii. Only a very small number equated the arguments correctly using $2\pi k$.

Use mathematical induction to prove that $(1-a)^n > 1-na$ for $\{n : n \in \mathbb{Z}^+,\ n \geqslant 2\}$ where $0 < a < 1$.

Markscheme

Let P_n be the statement: $(1-a)^n > 1-na$ for some $n \in \mathbb{Z}^+,\ n \geqslant 2$ where $0 < a < 1$ consider the case $n = 2$: $(1-a)^2 = 1-2a+a^2$ ***M1***

$> 1-2a$ because $a^2 < 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1-ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$> 1-(k+1)a+ka^2$ ***A1***

$> 1-(k+1)a \Rightarrow \mathrm{P}_{k+1}$ is true (**as** $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1-(k+1)a+ka^2$ ***A1***

$(1-a)^{k+1} > 1-(k+1)a \Rightarrow \mathrm{P}_{k+1}$ is true (**as** $ka^2 > 0$) ***R1***

THEN

P_2 is true P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true so P_n true for all $n > 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

Examiners report

[N/A]

Find the constant term in the expansion of $\left(x-\frac{2}{x}\right)^4\left(x^2+\frac{2}{x}\right)^3$.

Markscheme

$\left(x-\frac{2}{x}\right)^4 = x^4-8x^2+24-\frac{32}{x^2}+\frac{16}{x^4}$ ***(M1)(A1)***

$\left(x^2+\frac{2}{x}\right)^3 = x^6+6x^3+12+\frac{8}{x^3}$ ***(M1)(A1)***

Note: Accept unsimplified or uncalculated coefficients in the constant term

$= 24 \times 12$ ***(M1)(A1)***

$= 288$ ***A1***

[7 marks]

Examiners report

Many correct answers were seen, although most candidates used rather inefficient methods (e.g. expanding the brackets in multiple steps). In a very few cases candidates used the binomial theorem to obtain the answer quickly.

Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

Markscheme

METHOD 1

$1 + \mathrm{i}$ is a zero $\Rightarrow 1 - \mathrm{i}$ is a zero ***(A1)***

$1 - 2\mathrm{i}$ is a zero $\Rightarrow 1 + 2\mathrm{i}$ is a zero ***(A1)***

$(x - (1 - \mathrm{i}))\,(x - (1 + \mathrm{i})) = (x^2 - 2x + 2)$ ***(M1)A1***

$(x - (1 - 2\mathrm{i}))\,(x - (1 + 2\mathrm{i})) = (x^2 - 2x + 5)$ ***A1***

$p(x) = (x^2 - 2x + 2)(x^2 - 2x + 5)$ ***M1***

$= x^4 - 4x^3 + 11x^2 - 14x + 10$ ***A1***

$a = -4,\ b = 11,\ c = -14,\ d = 10$

[7 marks]

METHOD 2

$p(1 + \mathrm{i}) = -4 + (-2 + 2\mathrm{i})a + (2\mathrm{i})b + (1 + \mathrm{i})c + d$ ***M1***

$p(1 + \mathrm{i}) = 0 \Rightarrow \begin{cases} -4 - 2a + c + d = 0 \\ 2a + 2b + c = 0 \end{cases}$ ***M1A1A1***

$p(1 - 2\mathrm{i}) = -7 + 24\mathrm{i} + (-11 + 2\mathrm{i})a + (-3 - 4\mathrm{i})b + (1 - 2\mathrm{i})c + d$

$p(1 - 2\mathrm{i}) = 0 \Rightarrow \begin{cases} -7 - 11a - 3b + c + d = 0 \\ 24 + 2a - 4b - 2c = 0 \end{cases}$ ***A1***

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ -11 & -3 & 1 & 1 \\ 2 & -4 & -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \\ 7 \\ -24 \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \\ -14 \\ 10 \end{pmatrix}$$ ***M1A1***

$a = -4,\ b = 11,\ c = -14,\ d = 10$

[7 marks]

Examiners report

Most candidates attempted this question, using different approaches. The most successful approach was the method of complex conjugates and the product of linear factors. Candidates who used this method were in general successful whereas candidates who attempted direct substitution and separation of real and imaginary parts to obtain four equations in four unknowns were less successful because either they left the work incomplete or made algebraic errors that led to incorrect answers.

The 3rd term of an arithmetic sequence is 1407 and the 10th term is 1183.

a. Find the first term and the common difference of the sequence. [4]

b. Calculate the number of positive terms in the sequence. [3]

Markscheme

a. $u_1 + 2d = 1407$, $u_1 + 9d = 1183$ ***(M1)(A1)***

$u_1 = 1471$, $d = -32$ ***A1A1***

[4 marks]

b. $1471 + (n - 1)(-32) > 0$ ***(M1)***

$\Rightarrow n < \frac{1471}{32} + 1$

$n < 46.96\ldots$ ***(A1)***

so 46 positive terms ***A1***

[3 marks]

Examiners report

a. [N/A]

b. [N/A]

The sum of the second and third terms of a geometric sequence is 96.

The sum to infinity of this sequence is 500.

Find the possible values for the common ratio, r.

Markscheme

$ar + ar^2 = 96$ ***A1***

Note: Award ***A1*** for any valid equation involving a and r, *eg*, $\frac{a(1-r^3)}{1-r} - a = 96$.

$\frac{a}{1-r} = 500$ ***A1***

EITHER

attempting to eliminate a to obtain $500r(1 - r^2) = 96$ (or equivalent in unsimplified form) (***M1***)

OR

attempting to obtain $a = \frac{96}{r+r^2}$ and $a = 500(1 - r)$ (***M1***)

THEN

attempting to solve for r ***(M1)***

$r = 0.2 \left(= \frac{1}{5}\right)$ or $r = 0.885 \left(= \frac{\sqrt{97}-1}{10}\right)$ ***A1A1***

[6 marks]

Examiners report

Reasonably well done. Quite a number of candidates included a solution outside $-1 < r < 1$.

(a) Find the set of values of k for which the following system of equations has

no solution.

$$x + 2y - 3z = k$$
$$3x + y + 2z = 4$$
$$5x + 7z = 5$$

(b) Describe the geometrical relationship of the three planes represented by this system of equations.

Markscheme

(a)

$$\begin{pmatrix} 1 & 2 & -3 & k \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix} \quad \textbf{\textit{M1}}$$

$R_1 - 2R_2$

$$\begin{pmatrix} -5 & 0 & -7 & k-8 \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix} \quad \textbf{\textit{(A1)}}$$

$R_1 + R_3$

$$\begin{pmatrix} 0 & 0 & 0 & k-3 \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix} \quad \textbf{\textit{(A1)}}$$

Hence no solutions if $k \in \mathbb{R}, k \neq 3$ ***A1***

(b) Two planes meet in a line and the third plane is parallel to that line.

[5 marks]

Examiners report

Most candidates realised that some form of row operations was appropriate here but arithmetic errors were fairly common. Many candidates whose arithmetic was correct gave their answer as $k = 3$ instead of $k \neq 3$. Very few candidates gave a correct answer to (b) with most failing to realise that stating that there was no common point was not enough to answer the question.

A complex number z is given by $z = \frac{a+\mathrm{i}}{a-\mathrm{i}},\ a \in \mathbb{R}$.

(a) Determine the set of values of a such that

(i) z is real;

(ii) z is purely imaginary.

(b) Show that $|z|$ is constant for all values of a.

Markscheme

(a) $\frac{a+\mathrm{i}}{a-\mathrm{i}} \times \frac{a+\mathrm{i}}{a+\mathrm{i}}$ ***M1***

$= \frac{a^2-1+2a\mathrm{i}}{a^2+1} \left(= \frac{a^2-1}{a^2+1} + \frac{2a}{a^2+1}\mathrm{i}\right)$ ***A1***

(i) z is real when $a = 0$ ***A1***

(ii) z is purely imaginary when $a = \pm 1$ ***A1***

Note: Award ***M1A0A1A0*** for $\frac{a^2-1+2a\mathrm{i}}{a^2-1} \left(= 1 + \frac{2a}{a^2-1}\mathrm{i}\right)$ leading to $a = 0$ in (i).

[4 marks]

(b) **METHOD 1**

attempting to find either $|z|$ or $|z|^2$ by expanding and simplifying

eg $|z|^2 = \frac{\left(a^2-1\right)^2+4a^2}{\left(a^2+1\right)^2} = \frac{a^4+2a^2+1}{\left(a^2+1\right)^2}$ ***M1***

$= \frac{\left(a^2+1\right)^2}{\left(a^2+1\right)^2}$

$|z|^2 = 1 \Rightarrow |z| = 1$ ***A1***

METHOD 2

$|z| = \frac{|a+\mathrm{i}|}{|a-\mathrm{i}|}$ ***M1***

$|z| = \frac{\sqrt{a^2+1}}{\sqrt{a^2+1}} \Rightarrow |z| = 1$ ***A1***

[2 marks]

Total [6 marks]

Examiners report

Part (a) was reasonably well done. When multiplying and dividing by the conjugate of $a - \mathrm{i}$, some candidates incorrectly determined their denominator as $a^2 - 1$.

In part (b), a significant number of candidates were able to correctly expand and simplify $|z|$ although many candidates appeared to not understand the definition of $|z|$.

---

a. Find the term in x^5 in the expansion of $(3x + A)(2x + B)^6$. [4]

b. Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie. [4]

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six.

Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

Calculate the probability that five cookies are eaten.

Markscheme

a. $\left(A\binom{6}{5}2^5B+3\binom{6}{4}2^4B^2\right)x^5$ ***M1A1A1***

$=\left(192AB+720B^2\right)x^5$ ***A1***

[4 marks]

b. **METHOD 1**

$x=\frac{1}{6},\ A=\frac{3}{6}\left(=\frac{1}{2}\right),\ B=\frac{4}{6}\left(=\frac{2}{3}\right)$ ***A1A1A1***

probability is $\frac{4}{81}\ (=0.0494)$ ***A1***

METHOD 2

P (5 eaten) =P (M eats 1) P (N eats 4) + P (M eats 0) P (N eats 5) ***(M1)***

$=\frac{1}{2}\binom{6}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+\frac{1}{2}\binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$ ***(A1)(A1)***

$=\frac{4}{81}\ (=0.0494)$ ***A1***

[4 marks]

Examiners report

a. [N/A]

b. [N/A]

In a trial examination session a candidate at a school has to take 18 examination papers including the physics paper, the chemistry paper and the biology paper. No two of these three papers may be taken consecutively. There is no restriction on the order in which the other examination papers may be taken.

Find the number of different orders in which these 18 examination papers may be taken.

Markscheme

METHOD 1

consideration of all papers

all papers may be sat in 18! ways ***A1***

number of ways of positioning "pairs" of science subjects $=3!\times17!$ ***A1***

but this includes two copies of each "triple" ***(R1)***

number of ways of positioning "triplets" of science subjects $=3!\times16!$ ***A1***

hence number of arrangements is $18!-3!\times17!+3!\times16!$ ***M1A1***

$(=4.39\times10^{15})$

METHOD 2

consideration of all the non-science papers ***(M1)***

hence all non-science papers can be sat in 15! ways ***A1***

there are $16\times15\times14\ (=3360)$ ways of positioning the three science papers ***(M1)A1***

hence the number of arrangements is $16\times15\times14\times15!\ (=4.39\times10^{15})$ ***(M1)A1***

METHOD 3

consideration of all papers

all papers may be sat in 18! ways ***A1***

number of ways of positioning exactly two science subjects $= 3! \times 15! \times 16 \times 15$ ***M1A1***

number of ways of positioning “triplets” of science subjects $= 3! \times 16!$ ***A1***

hence number of arrangements is $18! - 3! \times 16! - 3! \times 15! \times 16 \times 15$ ***M1A1***

$(= 4.39 \times 10^{15})$

[6 marks]

Examiners report

[N/A]

The interior of a circle of radius 2 cm is divided into an infinite number of sectors. The areas of these sectors form a geometric sequence with common ratio k. The angle of the first sector is θ radians.

(a) Show that $\theta = 2\pi(1 - k)$.

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of k and of θ.

Markscheme

(a) the area of the first sector is $\frac{1}{2}2^2\theta$ ***(A1)***

the sequence of areas is $2\theta,\ 2k\theta,\ 2k^2\theta\ldots$ ***(A1)***

the sum of these areas is $2\theta(1 + k + k^2 + \ldots)$ ***(M1)***

$= \frac{2\theta}{1-k} = 4\pi$ ***M1A1***

hence $\theta = 2\pi(1 - k)$ ***AG***

Note: Accept solutions where candidates deal with angles instead of area.

[5 marks]

(b) the perimeter of the first sector is $4 + 2\theta$ ***(A1)***

the perimeter of the third sector is $4 + 2k^2\theta$ ***(A1)***

the given condition is $4 + 2k^2\theta = 2 + \theta$ ***M1***

which simplifies to $2 = \theta(1 - 2k^2)$ ***A1***

eliminating θ, obtain cubic in k: $\pi(1 - k)(1 - 2k^2) - 1 = 0$ ***A1***

or equivalent

solve for $k = 0.456$ and then $\theta = 3.42$ ***A1A1***

[7 marks]

Total [12 marks]

Examiners report

This was a disappointingly answered question.

Part(a) - Many candidates correctly assumed that the areas of the sectors were proportional to their angles, but did not actually state that fact.

Part(b) - Few candidates seem to know what the term ‘perimeter’ means.

Solve the following system of equations.

$$\log_{x+1} y = 2$$

$$\log_{y+1} x = \frac{1}{4}$$

Markscheme

$\log_{x+1} y = 2$

$\log_{y+1} x = \frac{1}{4}$

so $(x+1)^2 = y$ ***A1***

$(y+1)^{\frac{1}{4}} = x$ ***A1***

EITHER

$x^4 - 1 = (x+1)^2$ ***M1***

$x = -1$, not possible ***R1***

$x = 1.70$, $y = 7.27$ ***A1A1***

OR

1

$\left(x^2 + 2x + 2\right)^{\frac{1}{4}} - x = 0$ ***M1***

attempt to solve or graph of LHS ***M1***

$x = 1.70$, $y = 7.27$ ***A1A1***

[6 marks]

Examiners report

This question was well answered by a significant number of candidates. There was evidence of good understanding of logarithms. The algebra required to solve the problem did not intimidate candidates and the vast majority noticed the necessity of technology to solve the final equation. Not all candidates recognized the extraneous solution and there were situations where a rounded value of x was used to calculate the value of y leading to an incorrect solution.

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

Markscheme

the number of ways of allocating presents to the first child is $\binom{7}{3}$ $\left(\text{or } \binom{7}{2}\right)$ ***(A1)***

multiplying by $\binom{4}{2}$ $\left(\text{or } \binom{5}{3} \text{ or } \binom{5}{2}\right)$ ***(M1)(A1)***

Note: Award ***M1*** for multiplication of combinations.

$\binom{7}{3}\binom{4}{2} = 210$ ***A1***

[4 marks]

Examiners report

[N/A]

Two distinct roots for the equation $z^4 - 10z^3 + az^2 + bz + 50 = 0$ are $c + \mathrm{i}$ and $2 + \mathrm{i}d$ where $a,\ b,\ c,\ d \in \mathbb{R},\ d > 0$.

a. Write down the other two roots in terms of c and d. [1]

b. Find the value of c and the value of d. [6]

Markscheme

a. other two roots are $c - \mathrm{i}$ and $2 - \mathrm{i}d$ ***A1***

[1 mark]

b. **METHOD 1**

use of sum of roots ***(M1)***

$2c + 4 = 10$

$c = 3$ ***A1***

use of product of roots ***M1***

product is $(c + \mathrm{i})(c - \mathrm{i})(2 + \mathrm{i}d)(2 - \mathrm{i}d)$ ***A1***

$(c^2 + 1)(4 + d^2)\left[= 10(4 + d^2)\right] = 50$ ***A1***

Note: The line above can be awarded if they have used their value of c.

$d = 1$ ***A1***

METHOD 2

$z^4 - 10z^3 + az^2 + bz + 50 = (z^2 - 2cz + c^2 + 1)(z^2 - 4z + 4 + d^2)$ ***M1A1*** compare constant terms or coefficients of z^3 ***(M1)***

$4 + 2c = 10$

$(c^2 + 1)(4 + d^2) = 50$ ***A1***

$c = 3,\ d = 1$ ***A1A1***

[6 marks]

Examiners report

a. Most students using the sum and product of roots were able to work this problem through. There were many candidates who were attempting to multiply out, with varying degrees of success.

b. Most students using the sum and product of roots were able to work this problem through. There were many candidates who were attempting to multiply out, with varying degrees of success.

When $\left(1 + \frac{x}{2}\right)^2$, $n \in \mathbb{N}$, is expanded in ascending powers of x , the coefficient of x^3 is 70.

(a) Find the value of n .

(b) Hence, find the coefficient of x^2 .

Markscheme

(a) coefficient of x^3 is $\binom{n}{3}\left(\frac{1}{2}\right)^3 = 70$ ***M1(A1)***

$\frac{n!}{3!(n-3)!} \times \frac{1}{8} = 70$ ***(A1)***

$\Rightarrow \frac{n(n-1)(n-2)}{48} = 70$ ***(M1)***

$n = 16$ ***A1***

(b) $\binom{16}{2}\left(\frac{1}{2}\right)^2 = 30$ ***A1***

[6 marks]

Examiners report

Most candidates were able to answer this question well.

Find the constant term in the expansion of $\left(4x^2 - \frac{3}{2x}\right)^{12}$.

Markscheme

attempting a valid method to obtain the required term in the expansion ***(M1)***

Note: Valid methods include an attempt to expand, noting the behaviour of the powers of x, use of the general binomial expansion term, use of a ratio etc.

identifying the correct term ***(A1)***

$\binom{12}{8} \times 4^4 \times \left(-\frac{3}{2}\right)^8 \left(= 495 \times 4^4 \times \left(-\frac{3}{2}\right)^8\right)$ ***M1A1***

Note: Accept $\binom{12}{4}$.

Note: Award ***M1*** for the product of a binomial coefficient, a power of 4 and either a power of $-\frac{3}{2}$ or $\frac{3}{2}$.

$= 3\,247\,695$ ***A1***

[5 marks]

Examiners report

[N/A]

---

a. Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions. [5]

$$x - 3y + z = 3$$
$$x + 5y - 2z = 1$$
$$16y - 6z = k$$

b. Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. [7]

c. The plane $\div$ is parallel to both the line in part (b) and the line $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}$. [5]
Given that $\div$ contains the point $(1, 2, 0)$, show that the Cartesian equation of $\div$ is $16x + 24y - 11z = 64$.

d. The z-axis meets the plane $\div$ at the point P. Find the coordinates of P. [2]

e. Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane $\div$. [5]

Markscheme

a. in augmented matrix form $\left|\begin{matrix} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{matrix}\right|$

attempt to find a line of zeros *(M1)*

$r_2 - r_1 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{vmatrix}$ *(A1)*

$r_3 - 2r_2 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ *(A1)*

there is an infinite number of solutions when $k = -4$ ***R1***

there is no solution when

$k \neq -4,\ (k \in \mathbb{R})$ ***R1***

Note: Approaches other than using the augmented matrix are acceptable.

[5 marks]

b. using $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ and letting $\boldsymbol{z} = \lambda$ *(M1)*

$8y - 3\lambda = -2$

$\Rightarrow y = \frac{3\lambda - 2}{8}$ *(A1)*

$x - 3y + z = 3$

$\Rightarrow x - \left(\frac{9\lambda - 6}{8}\right) + \lambda = 3$ *(M1)*

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \frac{18 + \lambda}{8}$ *(A1)*

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$ *(M1)(A1)*

$r = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ ***A1***

Note: Accept equivalent answers.

[7 marks]

c. recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane *(A1)*

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ *(M1)*

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$ ***A1***

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$ *(M1)*

$\Rightarrow 16 + 48 = d$

$d = 64$ ***A1***

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ ***AG***

Note: Accept alternative methods using dot product.

[5 marks]

d. the plane crosses the z-axis when $x = y = 0$ *(M1)*

coordinates of P are $\left(0,\ 0,\ -\frac{64}{11}\right)$ ***A1***

Note: Award ***A1*** for stating $z = -\frac{64}{11}$.

Note: Accept. $\begin{pmatrix} 0 \\ 0 \\ -\frac{64}{11} \end{pmatrix}$

[2 marks]

e. recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector ***M1A1***
$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$ ***(A1)***
$\Rightarrow \theta = 42.8^\circ$ (0.747 radians) ***(A1)***
hence the angle between the line and the plane is 90° – 42.8° = 47.2° (0.824 radians) ***A1***
[5 marks]
Note: Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

Examiners report

a. Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

b. Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

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Find the sum of all the multiples of 3 between 100 and 500.

Markscheme

METHOD 1

$102 + 105 + \ldots + 498$ ***(M1)***

so number of terms = 133 ***(A1)***

EITHER

$= \frac{133}{2}(2 \times 102 + 132 \times 3)$ ***(M1)***

= 39900 ***A1***

OR

$= (102 + 498) \times \frac{133}{2}$ ***(M1)***

= 39900 ***A1***

OR

$\sum\limits_{n=34}^{166} 3n$ ***(M1)***

= 39900 ***A1***

METHOD 2

$500 \div 3 = 166.666...$ and $100 \div 3 = 33.333...$

$102 + 105 + \ldots + 498 = \sum\limits_{n=1}^{166} 3n - \sum\limits_{n=1}^{33} 3n$ ***(M1)***

$\sum\limits_{n=1}^{166} 3n = 41583$ ***(A1)***

$\sum\limits_{n=1}^{33} 3n = 1683$ ***(A1)***

the sum is 39900 ***A1***

[4 marks]

Examiners report

Most candidates got full marks in this question. Some mistakes were detected when trying to find the number of terms of the arithmetic sequence, namely the use of the incorrect value $n = 132$; a few interpreted the question as the sum of multiples between the 100th and 500th terms. Occasional application of geometric series was attempted.

The complex number $z = -\sqrt{3} + \mathrm{i}$.

a. Find the modulus and argument of z , giving the argument in degrees. [2]

b. Find the cube root of z which lies in the first quadrant of the Argand diagram, giving your answer in Cartesian form. [2]

c. Find the smallest positive integer n for which z^n is a positive real number. [2]

Markscheme

a. $\text{mod }(z) = 2, \ \arg(z) = 150^\circ$ ***A1A1***

[2 marks]

b. $z^{\frac{1}{3}} = 2^{\frac{1}{3}}(\cos 50^\circ + \mathrm{i} \sin 50^\circ)$ ***(M1)***

$= 0.810 + 0.965\mathrm{i}$ ***A1***

[2 marks]

c. we require to find a multiple of 150 that is also a multiple of 360, so by any method, ***M1***

$n = 12$ ***A1***

Note: Only award 1 mark for part (c) if $n = 12$ is based on $\arg(z) = -30$.

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

(a) (i) Find the sum of all integers, between 10 and 200, which are divisible by 7.

(ii) Express the above sum using sigma notation.

An arithmetic sequence has first term 1000 and common difference of -6 . The sum of the first n terms of this sequence is negative.

(b) Find the least value of n.

Markscheme

(a) (i) $n = 27$ ***(A1)***

METHOD 1

$S_{27} = \frac{14+196}{2} \times 27$ ***(M1)***

$= 2835$ ***A1***

METHOD 2

$S_{27} = \frac{27}{2}(2 \times 14 + 26 \times 7)$ ***(M1)***

$= 2835$ ***A1***

METHOD 3

$S_{27} \sum\limits_{n=1}^{27} 7 + 7n$ ***(M1)***

$= 2835$ ***A1***

(ii) $\sum\limits_{n=1}^{27} (7 + 7n)$ or equivalent ***A1***

Note: Accept $\sum\limits_{n=2}^{28} 7n$

[4 marks]

(b) $\frac{n}{2}(2000 - 6(n-1)) < 0$ ***(M1)***

$n > 334.333$

$n = 335$ ***A1***

Note: Accept working with equalities.

[2 marks]

Total [6 marks]

Examiners report

[N/A]

A geometric sequence has a first term of 2 and a common ratio of 1.05. Find the value of the smallest term which is greater than 500.

Markscheme

$2 \times 1.05^{n-1} > 500$ ***M1***

$n - 1 > \frac{\log\ 250}{\log\ 1.05}$ ***M1***

$n - 1 > 113.1675\ldots$ ***A1***

$n = 115$ ***(A1)***

$u_{115} = 521$ ***A1*** ***N5***

Note: Accept graphical solution with appropriate sketch.

[5 marks]

Examiners report

Many candidates misread the question and stopped at showing that the required term was the 115^{th}.

---

a. Write down the quadratic expression $2x^2+x-3$ as the product of two linear factors. [1]

b. Hence, or otherwise, find the coefficient of x in the expansion of $\left(2x^2+x-3\right)^8$. [4]

Markscheme

a. $2x^2+x-3=(2x+3)(x-1)$ ***A1***

Note: Accept $2\left(x+\frac{3}{2}\right)(x-1)$.

Note: Either of these may be seen in (b) and if so ***A1*** should be awarded.

[1 mark]

b. **EITHER**

$\left(2x^2+x-3\right)^8=(2x+3)^8(x-1)^8$ ***M1***

$=\left(3^8+8\left(3^7\right)(2x)+\ldots\right)\left((-1)^8+8(-1)^7(x)+\ldots\right)$ ***(A1)***

coefficient of $x=3^8\times8\times(-1)^7+3^7\times8\times2\times(-1)^8$ ***M1***

$=-17\,496$ ***A1***

Note: Under ft, final ***A1*** can only be achieved for an integer answer.

OR

$\left(2x^2+x-3\right)^8=\left(3-\left(x-2x^2\right)\right)^8$ ***M1***

$=3^8+8\left(-\left(x-2x^2\right)\left(3^7\right)+\ldots\right)$ ***(A1)***

coefficient of $x=8\times(-1)\times3^7$ ***M1***

$=-17\,496$ ***A1***

Note: Under ft, final ***A1*** can only be achieved for an integer answer.

[4 marks]

Examiners report

a. Many candidates struggled to find an efficient approach to this problem by applying the Binomial Theorem. A disappointing number of candidates attempted the whole expansion which was clearly an unrealistic approach when it is noted that the expansion is to the 8^{th} power. The fact that some candidates wrote down Pascal's Triangle suggested that they had not studied the Binomial Theorem in enough depth or in a sufficient variety of contexts.

b. Many candidates struggled to find an efficient approach to this problem by applying the Binomial Theorem. A disappointing number of candidates attempted the whole expansion which was clearly an unrealistic approach when it is noted that the expansion is to the 8^{th} power. The fact that some candidates wrote down Pascal's Triangle suggested that they had not studied the Binomial Theorem in enough depth or in a sufficient variety of contexts.

Consider the complex number $z = \frac{2+7\mathrm{i}}{6+2\mathrm{i}}$.

a. Express z in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{Q}$. [2]

b. Find the exact value of the modulus of z. [2]

c. Find the argument of z, giving your answer to 4 decimal places. [2]

Markscheme

a. $z = \frac{(2+7\mathrm{i})}{(6+2\mathrm{i})} \times \frac{(6-2\mathrm{i})}{(6-2\mathrm{i})}$ ***(M1)***

$= \frac{26+38\mathrm{i}}{40} = \left(\frac{13+19\mathrm{i}}{20} = 0.65 + 0.95\mathrm{i}\right)$ ***A1***

[2 marks]

b. attempt to use $|z| = \sqrt{a^2 + b^2}$ ***(M1)***

$|z| = \sqrt{\frac{53}{40}} \left(= \frac{\sqrt{530}}{20}\right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

c. **EITHER**

arg z = arg(2 + 7i) – arg(6 + 2i) ***(M1)***

OR

arg z = arctan$\left(\frac{19}{13}\right)$ ***(M1)***

THEN

arg z = 0.9707 (radians) (= 55.6197 degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The coefficient of x^2 in the expansion of $\left(\frac{1}{x} + 5x\right)^8$ is equal to the coefficient of x^4 in the expansion of $(a + 5x)^7,\ a \in \mathbb{R}$. Find the value of a.

Markscheme

METHOD 1

$^8C_r\left(\frac{1}{x}\right)^{8-r}(5x)^r = {}^8Cr(5)^r x^{2r-8}$ ***(M1)***

$r = 5$ ***(A1)***

$^8C_5 \times 5^5 = {}^7C_4 a^3 \times 5^4$ ***M1A1***

$56 \times 5 = 35a^3$

$a^3 = 8$ ***(A1)***

$a = 2$ ***A1***

METHOD 2

attempt to expand both binomials ***M1***

$175000x^2$ ***A1***

$21875a^3x^4$ ***A1***

$175000 = 21875a^3$ ***M1***

$a^3 = 8$ ***(A1)***

$a = 2$ ***A1***

[6 marks]

Examiners report

[N/A]

---

a. In an arithmetic sequence the first term is 8 and the common difference is $\frac{1}{4}$. If the sum of the first $2n$ terms is equal to the sum of the next n terms, find n. [9]

b. If $a_1,\ a_2,\ a_3,\ \ldots$ are terms of a geometric sequence with common ratio $r \neq 1$, show that $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \ldots + (a_n - a_{n+1})^2 = \frac{a_1^2(1-r)(1-r^{2n})}{1+r}$. [7]

Markscheme

a. $S_{2n} = \frac{2n}{2}\left(2(8) + (2n-1)\frac{1}{4}\right)$ ***(M1)***

$= n\left(16 + \frac{2n-1}{4}\right)$ ***A1***

$S_{3n} = \frac{3n}{2}\left(2 \times 8 + (3n-1)\frac{1}{4}\right)$ ***(M1)***

$= \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$S_{2n} = S_{3n} - S_{2n} \Rightarrow 2S_{2n} = S_{3n}$ ***M1***

solve $2S_{2n} = S_{3n}$

$\Rightarrow 2n\left(16 + \frac{2n-1}{4}\right) = \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$\left(\Rightarrow 2\left(16 + \frac{2n-1}{4}\right) = \frac{3}{2}\left(16 + \frac{3n-1}{4}\right)\right)$

(gdc or algebraic solution) ***(M1)***

$n = 63$ ***A2***

[9 marks]

b. $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \ldots$

$= (a_1 - a_1 r)^2 + (a_1 r - a_1 r^2)^2 + (a_1 r^2 - a_1 r^3) + \ldots$ ***M1A1***

$= [a_1(1-r)]^2 + [a_1 r(1-r)]^2 + \left[a_1 r^2(1-r)\right]^2 + \ldots + \left[a_1 r^{n-1}(1-r)\right]^2$ ***(A1)***

Note: This ***A1*** is for the expression for the last term.

$= a_1^2(1-r)^2 + a_1^2 r^2(1-r)^2 + a_1^2 r^4(1-r)^2 + \ldots + a_1^2 r^{2n-2}(1-r)^2$ ***A1***

$= a_1^2(1-r)^2(1 + r^2 + r^4 + \ldots + r^{2n-2})$ ***A1***

$= a_1^2(1-r)^2\left(\frac{1-r^{2n}}{1-r^2}\right)$ ***M1A1***

$= \frac{a_1^2(1-r)(1-r^{2n})}{1+r}$ ***AG***

[7 marks]

Examiners report

a. Many candidates were able to solve (a) successfully. A few candidates failed to understand the relationship between S_{2n} and S_{3n}, and hence did not obtain the correct equation. (b) was answered poorly by a large number of candidates. There was significant difficulty in forming correct general statements, and a general lack of rigor in providing justification.

b. Many candidates were able to solve (a) successfully. A few candidates failed to understand the relationship between S_{2n} and S_{3n}, and hence did not obtain the correct equation. (b) was answered poorly by a large number of candidates. There was significant difficulty in forming correct general statements, and a general lack of rigor in providing justification.

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

a. Find the first term and the common difference. [4]

b. Find the smallest value of *n* such that the sum of the first *n* terms is greater than 600. [3]

Markscheme

a. $S_n = \frac{n}{2}[2a + (n-1)d]$

$212 = \frac{16}{2}(2a + 15d) \quad (= 16a + 120d)$ ***A1***

n^{th} term is $a + (n-1)d$

$8 = a + 4d$ ***A1***

solving simultaneously: ***(M1)***

$d = 1.5,\ a = 2$ ***A1***

[4 marks]

b. $\frac{n}{2}[4 + 1.5(n-1)] > 600$ ***(M1)***

$\Rightarrow 3n^2 + 5n - 2400 > 0$ ***(A1)***

$\Rightarrow n > 27.4...,\ (n < -29.1...)$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ ***A1***
[3 marks]

Examiners report

a. This proved to be a good start to the paper for most candidates. The vast majority made a meaningful attempt at this question with many gaining the correct answers. Candidates who lost marks usually did so because of mistakes in the working. In part (b) the most efficient way of gaining the answer was to use the calculator once the initial inequality was set up. A small number of candidates spent valuable time unnecessarily manipulating the algebra before moving to the calculator.

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The first term and the common ratio of a geometric series are denoted, respectively, by *a* and *r* where a , $r \in \mathbb{Q}$. Given that the third term is 9 and the sum to infinity is 64, find the value of *a* and the value of *r* .

Markscheme

we are given that $ar^2 = 9$ and $\frac{a}{1-r} = 64$ ***A1***

dividing, $r^2(1-r) = \frac{9}{64}$ ***M1***

$64r^3 - 64r^2 + 9 = 0$ ***A1***

$r = 0.75,\ a = 16$ ***A1A1***

[5 marks]

Examiners report

[N/A]

Find the coefficient of x^{-2} in the expansion of $(x-1)^3\left(\frac{1}{x} + 2x\right)^6$.

Markscheme

expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ ***A1***

expanding $\left(\frac{1}{2} + 2x\right)^6$ gives

$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$ ***(M1)A1A1***

Note: Award ***(M1)*** for an attempt at expanding using binomial.

Award ***A1*** for $\frac{60}{x^2}$.

Award ***A1*** for $\frac{12}{x^4}$.

$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$ ***(M1)***

Note: Award ***(M1)*** only if both terms are considered.

therefore coefficient x^{-2} is -96 ***A1***

Note: Accept $-96x^{-2}$

Note: Award full marks if working with the required terms only without giving the entire expansion.

[6 marks]

Examiners report

[N/A]

A bank offers loans of $\$P$ at the beginning of a particular month at a monthly interest rate of I . The interest is calculated at the end of each month and added to the amount outstanding. A repayment of $\$R$ is required at the end of each month. Let $\$S_n$ denote the amount outstanding immediately after the n^{th} monthly repayment.

a. (i) Find an expression for S_1 and show that [7]

$$S_2 = P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \left(1 + \frac{I}{100}\right)\right).$$

(ii) Determine a similar expression for S_n . Hence show that

$$S_n = P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1 + \frac{I}{100}\right)^n - 1\right)$$

b. Sue borrows $5000 at a monthly interest rate of 1 % and plans to repay the loan in 5 years (*i.e.* 60 months). [6]

(i) Calculate the required monthly repayment, giving your answer correct to two decimal places.

(ii) After 20 months, she inherits some money and she decides to repay the loan completely at that time. How much will she have to repay, giving your answer correct to the nearest $?

Markscheme

a. (i) $S_1 = P\left(1 + \frac{I}{100}\right) - R$ ***A1***

$S_2 = P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \frac{I}{100}\right) - R$ ***M1A1***

$= P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \left(1 + \frac{I}{100}\right)\right)$ ***AG***

(ii) extending this,

$S_n = P\left(1 + \frac{I}{100}\right)^n - R\left(1 + \left(1 + \frac{I}{100}\right) + \ldots + \left(1 + \frac{I}{100}\right)^{n-1}\right)$ ***M1A1***

$= P\left(1 + \frac{I}{100}\right)^n - \frac{R\left(\left(1+\frac{I}{100}\right)^n - 1\right)}{\frac{I}{100}}$ ***M1A1***

$= P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1 + \frac{I}{100}\right)^n - 1\right)$ ***AG***

[7 marks]

b. (i) putting $S_{60} = 0,\ P = 5000,\ I = 1$ ***M1***

$5000 \times 1.01^{60} = 100R\left(1.01^{60} - 1\right)$ ***A1***

$R = (\$)111.22$ ***A1***

(ii) $n = 20,\ P = 5000,\ I = 1,\ R = 111.22$ ***M1***

$S_{20} = 5000 \times 1.01^{20} - 100 \times 111.22(1.01^{20} - 1)$ ***A1***

$= (\$)3652$ ***A1***

which is the outstanding amount

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

In the arithmetic series with n^{th} term u_n , it is given that $u_4 = 7$ and $u_9 = 22$.

Find the minimum value of n so that $u_1 + u_2 + u_3 + \ldots + u_n > 10\,000$.

Markscheme

$u_4 = u_1 + 3d = 7$, $u_9 = u_1 + 8d = 22$ ***A1A1***

Note: $5d = 15$ gains both above marks

$u_1 = -2$, $d = 3$ ***A1***

$S_n = \frac{n}{2}(-4 + (n-1)3) > 10\,000$ ***M1***

$n = 83$ ***A1***

[5 marks]

Examiners report

This question was well answered by most candidates. A few did not realise that the answer had to be an integer.

Solve the simultaneous equations

$$\ln \frac{y}{x} = 2$$

$$\ln x^2 + \ln y^3 = 7.$$

Markscheme

METHOD 1

$\ln \frac{y}{x} = 2 \Rightarrow \ln x + \ln y = 2$ ***A1***

$\ln x^2 + \ln y^3 = 7 \Rightarrow 2\ln x + 3\ln y = 7$ ***(M1)A1***

attempting to solve for x and y $\left(\text{to obtain } \ln x = \frac{1}{5} \text{ and } \ln y = \frac{11}{5}\right)$ ***(M1)***

$x = \mathrm{e}^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = \mathrm{e}^{\frac{11}{5}}\ (= 9.03)$ ***A1***

METHOD 2

$\ln \frac{y}{x} = 2 \Rightarrow y = \mathrm{e}^2 x$ ***A1***

$\ln x^2 + \ln \mathrm{e}^6 x^3 = 7$ ***(M1)A1***

attempting to solve for x ***(M1)***

$x = \mathrm{e}^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = \mathrm{e}^{\frac{11}{5}}\ (= 9.03)$ ***A1***

METHOD 3

$\ln \frac{y}{x} = 2 \Rightarrow y = \mathrm{e}^2 x$ ***A1***

$\ln x^2 + \ln y^3 = 7 \Rightarrow \ln(x^2 y^3) = 7$ ***A1***

$x^2 y^3 = \mathrm{e}^7$ ***(M1)***

substituting $y = e^2 x$ into $x^2 y^3 = \mathrm{e}^7$ (to obtain $\mathrm{e}^6 x^5 = \mathrm{e}^7$) ***M1***

$x = \mathrm{e}^{\frac{1}{5}}\ (= 1.22)$ ***A1***

$y = \mathrm{e}^{\frac{11}{5}}\ (= 9.03)$ ***A1***

[6 marks]

Examiners report

Reasonably well done. Candidates who did not obtain the correct solution generally made an error when attempting to apply logarithmic or exponential laws and hence made erroneous substitutions.

Let $f(x) = \ln x$. The graph of *f* is transformed into the graph of the function *g* by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the *x*-axis.

Find an expression for $g(x)$, giving your answer as a single logarithm.

Markscheme

$h(x) = f(x-3) - 2 = \ln(x-3) - 2$ ***(M1)(A1)***

$g(x) = -h(x) = 2 - \ln(x-3)$ ***M1***

Note*:* Award **M1** only if it is clear the effect of the reflection in the *x*-axis:

the expression is correct ***OR***
there is a change of signs of the previous expression ***OR***
there's a graph or an explanation making it explicit

$= \ln \mathrm{e}^2 - \ln(x-3)$ ***M1***

$= \ln\left(\frac{\mathrm{e}^2}{x-3}\right)$ ***A1***

[5 marks]

Examiners report

This question was well attempted but many candidates could have scored better had they written down all the steps to obtain the final expression. In some cases, as the final expression was incorrect and the middle steps were missing, candidates scored just 1 mark. That could be a consequence of a small mistake, but the lack of working prevented them from scoring at least all method marks. Some candidates performed the transformations well but were not able to use logarithms properties to transform the answer and give it as a single logarithm.

Find the value of *k* such that the following system of equations does not have a unique solution.

$$kx + y + 2z = 4$$

$$-y + 4z = 5$$

$$3x + 4y + 2z = 1$$

Markscheme

METHOD 1

determinant = 0 ***M1***

$k(-2-16) - (0-12) + 2(0+3) = 0$ ***(M1)(A1)***

$-18k + 18 = 0$ ***(A1)***

$k = 1$ ***A1***

METHOD 2

writes in the form

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$ (or attempts to solve simultaneous equations) ***(M1)***

Having two 0's in first column (obtaining two equations in the same two variables) ***M1***

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 18k-18 & 21k-27 \end{pmatrix}$$ (or isolating one variable) ***A1***

Note: The ***A1*** is to be awarded for the $18k - 18$. The final column may not be seen.

$k = 1$ ***(M1)A1***

[5 marks]

Examiners report

Candidates who used the determinant method usually obtained full marks. Few students used row reduction and of those the success was varied. However, many candidates attempted long algebraic methods, which frequently went wrong at some stage. Of those who did work through to correctly isolate one variable, few were able to interpret the resultant value of k .

A. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$, [8]

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

B. (a) Using integration by parts, show that $\int e^{2x} \sin x \mathrm{d}x = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$. [17]

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}e^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

Markscheme

A. prove that $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$

for $n = 1$

$\text{LHS} = 1,\ \text{RHS} = 4 - \frac{1+2}{2^0} = 4 - 3 = 1$

so true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n = k+1$

LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***

$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all n $(\in \mathbb{Z}^+)$. ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

B. (a)

METHOD 1

$\int e^{2x}\sin x dx = -\cos x e^{2x} + \int 2e^{2x}\cos x dx$ ***M1A1A1***

$= -\cos x e^{2x} + 2e^{2x}\sin x - \int 4e^{2x}\sin x dx$ ***A1A1***

$5\int e^{2x}\sin x dx = -\cos x e^{2x} + 2e^{2x}\sin x$ ***M1***

$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$ ***AG***

METHOD 2

$\int \sin x e^{2x} dx = \frac{\sin x e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} dx$ ***M1A1A1***

$= \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4} dx$ ***A1A1***

$\frac{5}{4}\int e^{2x}\sin x dx = \frac{e^{2x}\sin x}{2} - \frac{\cos x e^{2x}}{4}$ ***M1***

$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$ ***AG***

[6 marks]

(b)

$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x}\sin x dx$ ***M1A1***

$\arcsin y = \frac{1}{5}e^{2x}(2\sin x - \cos x)(+C)$ ***A1***

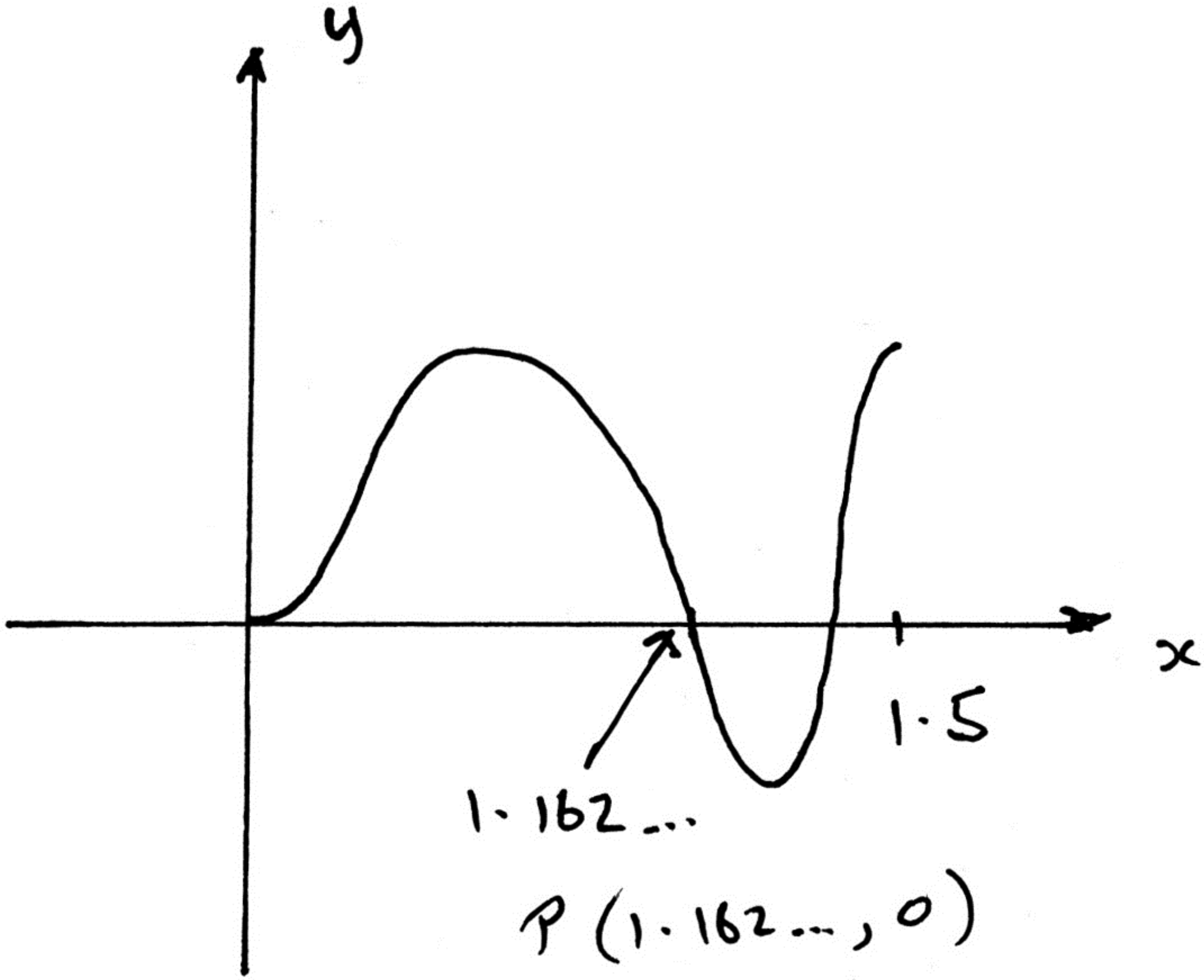
when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin\left(\frac{1}{5}e^{2x}(2\sin x - \cos x) + \frac{1}{5}\right)$ ***A1***

[5 marks]

(c)

(i)



A1

P is (1.16, 0) *A1*

Note: Award *A1* for 1.16 seen anywhere, *A1* for complete sketch.

Note: Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162\ldots} \pi y^2 \mathrm{d}x$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Examiners report

A. Part A: Given that this question is at the easier end of the 'proof by induction' spectrum, it was disappointing that so many candidates failed to score full marks. The $n = 1$ case was generally well done. The whole point of the method is that it involves logic, so 'let n = k' or 'put n = k', instead of 'assume ... to be true for n = k', gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

B. Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

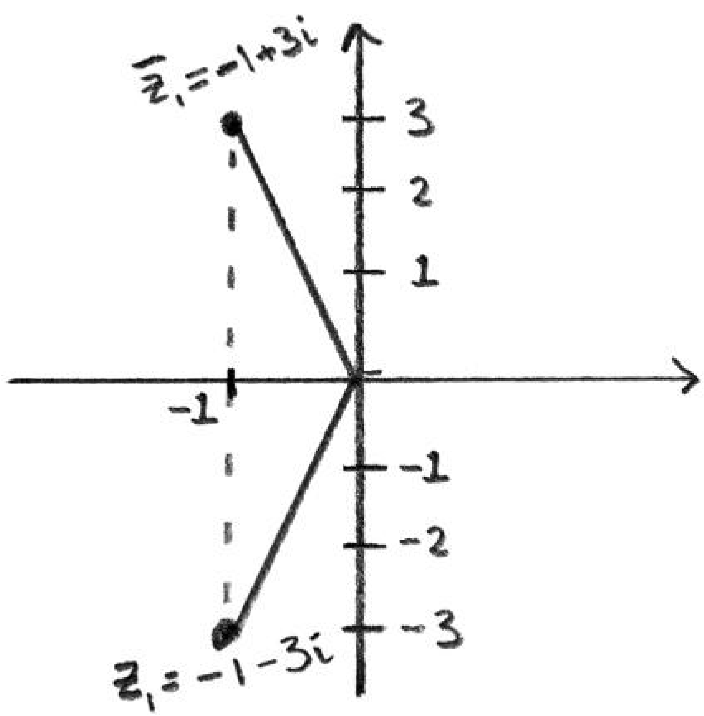
Consider the equation $z^3 + az^2 + bz + c = 0$, where a , b, $c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3\mathrm{i}$, find

(a) the other two roots;

(b) a , b and c .

Markscheme

(a) one root is $-1 - 3\mathrm{i}$ ***A1***



distance between roots is 6, implies height is 3 ***(M1)A1***

EITHER

$-1 + 3 = 2 \Rightarrow$ third root is 2 ***A1***

OR

$-1-3=-4 \Rightarrow$ third root is 4 ***A1***

(b) **EITHER**

$z-(-1+3\mathrm{i}) \quad z-(-1-3\mathrm{i}) \quad (z-2)=0$ ***M1***

$\Rightarrow \left(z^2+2z+10\right)(z-2)=0$ ***(A1)***

$z^2+6z-20=0$ ***A1***

$a=0$, $b=6$ and $c=20$

OR

$z-(-1+3\mathrm{i}) \quad z-(-1-3\mathrm{i}) \quad (z+4)=0$ ***M1***

$\Rightarrow \left(z^2+2z+10\right)(z+4)=0$ ***(A1)***

$z^2+6z^2+18z+40=0$ ***A1***

$a=6$, $b=18$ and $c=40$

[7 marks]

Examiners report

Most students were able to state the conjugate root, but many were unable to take the question further. Of those that then recognised the method, the question was well answered.